The Magic Circle!

Ted Ostrander and Arthur Wiebe

A simple circular piece of paper can teach a surprising number of geometric concepts! Your students will be continuously involved as they fold and unfold, observe, compare, and draw conclusions. By being engaged in a hands-on activity, students will experience geometric concepts in a real world context.

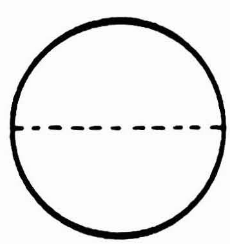
For this activity cut out circles about 25 centimeters (9-10 inches) in diameter out of newspaper or similar material. Each student and the teacher need a circle in order to act upon, feel, see, and discuss each step.

A circle is formed on a piece of paper by a line that is the same distance from a specific point called the center of the circle.

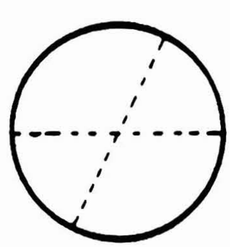
In this article the hands-on activities are indicated by bold type. The answers to the discussion questions are given in parentheses. It is recommended that each question be thoroughly discussed so students learn new terms, concepts, and facts.

For use with students, take a few steps at a time. The sequence is presented here in its entirety only for the purpose of showing the broad scope that is possible. The first time through, it might be advisable just to do the folding as outlined in bold type. This could then be followed by studying the questions after each fold.

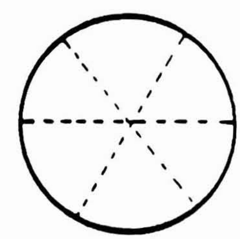
1. **Using the paper circle, fold the circle in half.**
   a. What is this new shape called? (Semi-circle)
   b. What is its straight edge called? (Diameter)
   c. Define a diameter. (The longest line segment with endpoints on the circle)
   d. How much of the area of the circle is in the semi-circle? (One-half).
   e. How can you find the center of the circle? (Fold another different semi-circle).



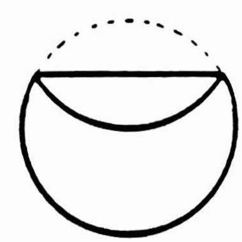
2. **Open the circle. Fold a second semi-circle. Mark the point where the diameters intersect.**
   a. Open the circle. What can you say about the two diameters? (They intersect at the center of the circle, bisect each other and are congruent.)
   b. What is the line segment that is one-half the diameter called? (Radius).
   c. The plural of radius is radii. How many radii can you see? (Four)
   d. **A sector of a circle is formed by two radii and the arc connecting an endpoint of each. How many sectors do you see? (Four).**
   e. **How do the shape and size of opposite sectors compare? (They are congruent: same size and same shape.)**
   f. **How does the measure of opposite angles compare?** (It is the same. The angles are congruent.)
   g. What shape do any two adjacent sectors form? (Semi-circle).
   h. What is the sum of any two adjacent arcs? (180 degrees or a semi-circle.)
   i. If two adjacent sectors are combined, how does their area compare with the area of the circle? (It is one-half the area of the circle.)



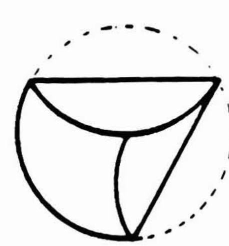
3. **Fold a third semi-circle.**
   a. What can you say about the three diameters? (They are congruent and intersect in the center of the circle.)
   b. How many radii do you see? (Six)
   c. Compare the radii. (They are congruent.)
   d. How many sectors do you see? (Six)
   e. What shape do any three adjacent sectors form? (Semi-circle).
   f. Complete this statement: "Any fold that passes through the center of the circle forms a ______? (Diameter)
   g. How many pairs of opposite angles are there? (Three)
   h. How does the measures of opposite angles compare? (They are the same.)



4. **Open up the circle and fold one edge of the circle to the center and fold.**
   a. What is the straight line segment formed by the fold called? (A chord. All folds with endpoints on the circle not passing through the center are called chords of the circle.)
   b. What is a special name for the arc that remains? (Major arc since its measure is more than 180 degrees.)



5. **Make a second fold to the center so that the end of the new chord meets an end of the first chord. Crease. The figure now resembles an ice cream cone.**
   a. What elements form its boundary? (Two chords and an arc)



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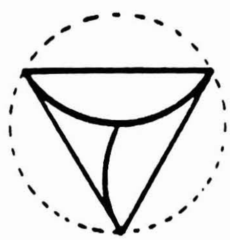
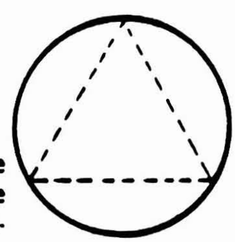
b. What part of the circle is in the intercepted arc? (1/3)
c. What is the measure of the arc? (120 degrees)
d. What is the angle formed by the two chords called? (Inscribed angle)
e. What is the measure of the inscribed angle? (60 degrees)
f. How does the measure of the angle compare with the measure of the arc? (It is 1/2 the measure of the intercepted arc.)

6. Make a third fold to the center so that the ends of the new chord meet the remaining ends of the previous two chords. Crease.



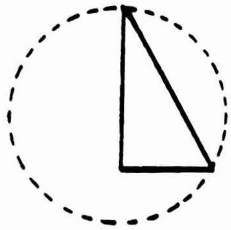
a. What is the shape that is formed called? (Equilateral triangle)
b. How do the measures of the sides of the equilateral triangle compare? (They are congruent.)
c. How do the angles of the equilateral triangle compare? (They are congruent.)
d. Is the area of the equilateral triangle more or less than one-half the area of the circle? (Less than one-half since the triangle is completely double covered everywhere and partially triple covered.)
e. Given that the sum of the measures of angles of a triangle is 180 degrees, what is the measure of each of the angles? (60 degrees)
f. In this figure a vertex is formed where two chords meet. How many vertices do you see? (Three)

7. Open up the circle. Note that the equilateral triangle is inscribed in the circle; that is, each vertex is on the circle.



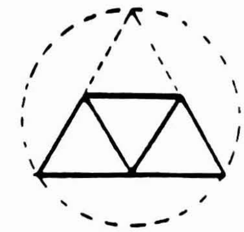
How does the area of the equilateral triangle compare with the area of the circle outside the triangle? (It is less than one-half since the outside will cover all of the triangle at least twice and some of it three times.)

8. Fold back into an equilateral triangle. Find the midpoint of one side. Make a fold passing through this midpoint and the opposite vertex.



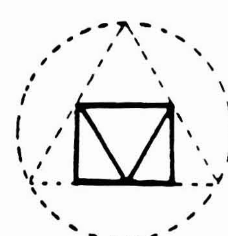
a. What type of triangle is formed? (Right triangle)
b. What is the measure of each of its angles? (90, 60, and 30 degrees.)
c. The longest side of this right triangle is called the hypotenuse. The shorter two sides are called legs. How do the lengths of the legs compare? (The shorter is one-half the hypotenuse or longest side.)
d. What is the measure of the angle opposite the hypotenuse? (90 degrees)
e. What is the measure of the angle opposite the shorter leg? (30 degrees)
f. What is the measure of the angle opposite the longer leg? (60 degrees)
g. How does the area of the right triangle compare with that of the equilateral triangle? (It is one-half that of the equilateral triangle.)

9. Open up into the equilateral triangle. Bring one vertex of the triangle to the opposite mid-point of a side and crease well.



a. What type of figure is formed? (Isosceles trapezoid)
b. Compare one set of opposite sides of the trapezoid. What do you observe? (In one set the sides are congruent but not parallel. In the other the sides are parallel but not congruent: one is twice the length of the other.)
c. Compare the second set of opposite sides of the trapezoid. What do you observe? (The other condition above).
d. How many triangles do you see? (Three are visible and and a fourth is under the center triangle.)
e. How do the four triangles compare? (They are congruent.)
f. How does the area of the trapezoid compare with that of the first large equilateral triangle? (It is three-fourths that of the triangle.)
g. Explain your reasoning. (Since the first triangle has been divided into four congruent triangles and three form the trapezoid, the area of the trapezoid is 3/4 that of the first triangle.)
h. How does the area of the orginal equilateral triangle compare with that of the trapezoid? (It is 4/3 that of the trapezoid.)
i. How does the area of one of the four equilateral triangles compare with that of the trapezoid? (They are 1/3 the area of the trapezoid.)
j. How does the area of one of the four triangles compare with that of the large equilateral triangle? (It has 1/4 the area.)
k. How does the perimeter of the trapezoid compare with that of the large equilateral triangle? (It is 5/6 as long.)

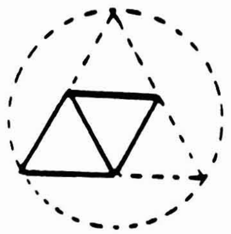
10. Fold the second and third vertices to the same midpoint.



a. What type of figure is formed? (Rectangle)
b. How does the height (or width) of this rectangle compare with the height of the original triangle? (1/2 as high)
c. How does the length of this rectangle compare with length of the base of the original triangle? (1/2 as long)

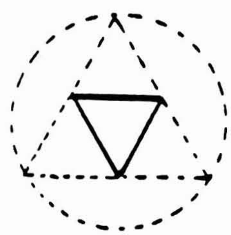
d. If the base of the triangle is b and the height of the triangle is h, what are the dimensions of this rectangle? ($\frac{1}{2}$b by $\frac{1}{2}$h)
e. What is the area of the rectangle? ($\frac{1}{4}$bh)
f. What, then, is the area of the original triangle? (Twice as much or $\frac{1}{2}$bh)
g. Note that the vertices of the triangle are all together. What is the sum of the measure of all three vertices? (A straight angle or 180 degrees)

11. With the trapezoid, fold one of the outside triangles over the center triangle.



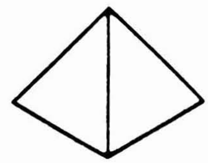
a. What figure is formed? (Rhombus)
b. How does the measure of the sides of the rhombus compare? (They are equal.)
c. What can you say about opposite sides? (They are congruent and parallel.)
d. How do opposite angles compare? (They are congruent.)
e. How many small triangles do you see? (Two)
f. The area of the rhombus is what part of the area of the large triangle? (One-half)
g. The area of the rhombus is what part of the area of the trapezoid? (Two-thirds)
h. How does the area of the rhombus compare with that of the small triangles? (It is twice as large.)
i. Compare the area of the large triangle to that of the rhombus. (It is twice as large.)
j. Compare the area of the trapezoid to that of the rhombus. (It is $\frac{3}{2}$ as large.)
k. How does the perimeter of the rhombus compare with that of the large equilateral triangle? (It is $\frac{4}{6}$ or $\frac{2}{3}$ as long.)
l. How does the perimeter of the rhombus compare with that of the trapezoid? (It is $\frac{4}{5}$ as long.)
m. How does the perimeter of the large equilateral triangle compare with that of the rhombus? (It is $\frac{3}{2}$ or 1.5 times as long.)
n. How does the perimeter of the trapezoid compare with that of the rhombus? (It is $\frac{5}{4}$ or 1.25 times as long.)
A rhombus is a subset of the set of parallelograms.

12. Fold the remaining outside triangle over the center triangle.



a. What figure is formed? (Equilateral triangle)
b. How does the area of this triangle compare with that of the rhombus? (It is one-half as large.)
c. How does the area of this triangle compare with the right triangle folded earlier? (The area of the triangle is $\frac{1}{2}$ that of the right triangle.)
d. How does the perimeter compare with that of the rhombus? (It is $\frac{3}{4}$ that of the rhombus.)
e. How does the perimeter of the triangle compare with that of the trapezoid? (It is $\frac{3}{5}$ that of the trapezoid.)
f. How does the perimeter of the triangle compare with that of the large equilateral triangle? (It is $\frac{1}{2}$ that of the large triangle.)
g. How does the perimeter of the trapezoid compare with that of the small equilateral triangle? (It is $\frac{5}{3}$ or $1\frac{2}{3}$ as long.)
h. How does the perimeter of the rhombus compare with that of the small triangle? (It is $\frac{4}{3}$ or $1\frac{1}{3}$ as long.)
i. How does the perimeter of the large equilateral triangle compare with that of the small triangle? (It is twice as long.)

13. Open the figure to the original triangle. Fold the original triangle so the vertices meet to form a three-dimensional pyramid.

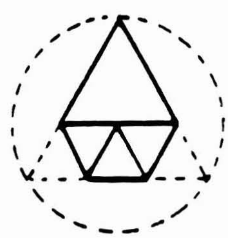


a. What is this figure called? (Tetrahedron. Tetra means four. It has four faces.)
b. What is the surface area of the tretrahedron if the area of the original triangle is 1? (Also 1)
c. What is the surface area of the tetrahedron if the area of one of the small triangles is 1? (4)
d. What is the surface area of the tetrahedron if the area of the trapezoid is 1? ($\frac{4}{3}$)
e. What is the surface area of the tetrahedron if the area of the rhombus is 1? (2)

14. Open the figure to the original triangle. Fold one vertex to the center of the circle. (Make the folds in Steps 13, 14, and 15; then return to the condition in this step. The triangles will be easier to visualize.)

a. What new figure is formed? (Isosceles trapezoid) Explain the answer.
b. How does the area of this trapezoid compare with the original triangle? (It is $\frac{8}{9}$ as large.)
c. How does the perimeter of this trapezoid compare with that of the original triangle? (It is $\frac{8}{9}$ as long.)
d. How does the short base of the trapezoid compare with the long base? (It is $\frac{1}{3}$ as long.)

15. Fold a second vertex to the center.



a. What is the new figure called? (Pentagon)
b. How does the area of the pentagon compare with that of the original triangle? (It is $\frac{7}{9}$ as large.)

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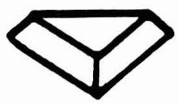
c. How does the perimeter of the pentagon compare with that of the original triangle? (It is $7/9$ as long.)
d. How does the area of the pentagon compare with that of the isosceles trapezoid in previous case? (It is $7/8$ as large.)
e. How does the perimeter of the pentagon compare with that of the isosceles trapezoid? (It is $7/8$ as long.)

16. Fold the third vertex to the center of the circle.

a. What is the new figure called? (Regular hexagon)
A regular hexagon has all sides and interior angles congruent.
b. How many small triangles do you see? (Six)
c. How does the area of the hexagon compare with that of the original triangle? (It is $6/9$ or $2/3$ as large.)
d. How does the perimeter of the hexagon compare with that of the original triangle? (It is $6/9$ or $2/3$ as long.)
e. How does the area of the hexagon compare with that of the pentagon? (It is $6/7$ as large.)
f. How does the area of the hexagon compare with the trapezoid in this sequence? (It is $6/8$ or $3/4$ as large.)
g. How does the perimeter of the hexagon compare with that of the pentagon? (It is $6/7$ as long.)
h. How does the perimeter of the hexagon compare with that of the trapezoid in this sequence? (It is $6/8$ or $3/4$ as long.)
i. How does the area of the hexagon compare with that of the first trapezoid, not the one in this sequence? (The area of the hexagon is $2/3$ or $8/12$ that of the triangle and the trapezoid is $3/4$ or $9/12$ that of the triangle. Therefore, the area of the triangle is $8/9$ that of the trapezoid.)
j. How does the area of the hexagon compare with that of the rhombus? (The area of the hexagon is $8/12$ that of the triangle and the area of the rhombus is $1/2$ or $6/12$ that of the triangle. Therefore, the area of the hexagon is $8/6$ or $4/3$ that of the rhombus.)

It is clear at this stage that additional rather difficult comparisons could be examined. These are left up to the discretion of the teacher.

17. Open up the triangle. Tuck one of the small triangles at one of the vertices into the small triangle at one of the other vertices. Then tuck the triangle at the remaining vertex underneath the other two to form a three-dimensional figure.



a. What is the new figure called? (It is a truncated tetrahedron. Truncated means it has a part that has been cut off.)
b. **If the original triangle had an area of 1, what is the area of (1) the bottom base?** ($1/4$)
(2) the top base? ($1/9$)
(3) a side face or wall? ($1/4 - 1/9$) (Note that together one side face and the top base are congruent to the bottom base.)
c. What is the total surface area of the truncated tetrahedron?
$[1/4 + 1/9 + 3(1/4 - 1/9)] = [9/36 + 4/36 + 3(9/36 - 4/36)] =$
$[13/36 + 3(5/36)] = 28/36 = 7/9$.
d. Remembering Polya's advice that "it is better to work one problem five ways than to work five problems one way" can you think of another way to solve this problem? (When the truncated tetrahedron was formed, we hid exactly two of the small triangles. All the rest of the original triangle is to be found in the bases and side faces. Each of the hidden triangles had $1/9$ the area of the original triangle. Therefore, $2/9$ of that area was lost, leaving $7/9$.)

18. Build a truncated tetrahedron that will just fit on top of this truncated tetrahedron.

The Magic Circle Revisited

Dr. Arthur J. Wiebe

The magic circle contains additional problem solving challenges!

We can revisit the magic circle and make quantitative computations of relative areas and perimeters by assigning a convenient value to the measure of the radius of the circle. For the following study we assign the value of 1 unit to the measure of the radius. All results, of course, are rounded off.

1. **If the measure of the radius is 1 unit,**
   a. **what is the measure of the circumference?** (Diameter times pi or approximately 6.28 units)
   b. **what is the measure of the area of the circle?** (Radius squared times pi or 3.14 square units)
2. **What conclusions can we draw from studying Figure 1.**

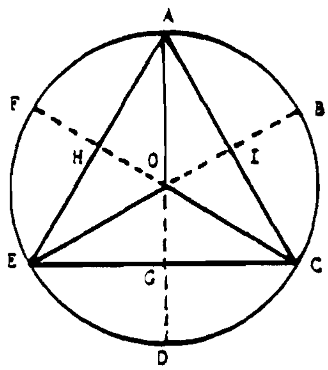


Figure 1

In Figure 1, lines are drawn from each vertex of the equilateral triangle to the circle passing through the midpoint of the opposite side of the triangle. From this we can make a number of observations. Note, these are observations and not formal proofs.

Line segments such as AG serve four functions: they are the angle bisectors, perpendicular bisectors of the sides, altitudes, and medians. This is true only in equilateral triangles.

The circle and the triangle have both been divided into six congruent regions by these lines. Therefore, the six central angles are congruent and each has a measure of 60 degrees; AG, CH, and EI are congruent; OC, OA, and OE are also congruent; therefore, OG, OH, and OI are congruent.

We know that the medians of a triangle intersect at a point two-thirds the distance from the vertex to the opposite side. Therefore, OC is twice the length of OH. Hence, OC is also twice the length of OG. If we assign the measure of 1 to the radius OC, then OG has the measure of ½.

3. **What is the measure of the perimeter of the inscribed triangle?** ($3\sqrt{3}$ or about 5.196 units)

In figures 1 and 2, OC is a radius and has an assigned measure of 1 unit and hence OG has a measure of ½ unit. We also know that AG is perpendicular to EC and that OCG is a right triangle. Using the Theorem of Pythagoras (the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse) we find that the measure of GC is $\sqrt{3}/2$ units and EC a measure of $\sqrt{3}$ units.

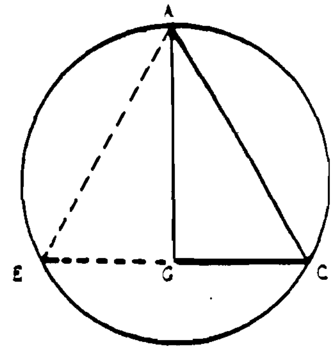


Figure 2

Therefore, the perimeter of the equilateral triangle then has a measure of $3\sqrt{3}$ or approximately 5.196 units.

4. **What is the measure of the area of the inscribed triangle?**

Since the measure of a side is $\sqrt{3}$ from the above, our task is simplified. The base of triangle ADF has a measure of $\sqrt{3}$. The altitude has a measure of 1.5. Can you explain why? (OA has a measure of 1 and OG a measure of ½)

Therefore, the measure of the area of the triangle is $½(\sqrt{3})(1.5)$ square units or $.75\sqrt{3}$ or 1.299 square units.

5. **Approximately what per cent of the measure of the**
   a. **circumference of the circle is the measure of the perimeter of the triangle?** (5.196/6.28=82.7%)
   b. **area of the circle is the measure of the area of triangle?** (1.299/3.14=41.4%)
6. **Consider the sector AOB, which is one-sixth of the circle,**
   a. **what is the measure of its boundary?** (1+1+6.28/6=3.047 square units)
   b. **what is the measure of its area?** (3.14/6=.52 square units)

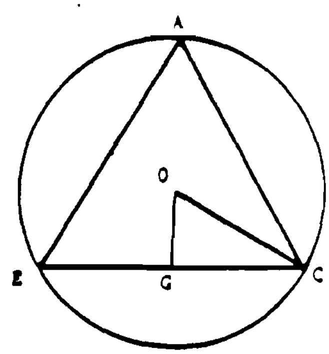


Figure 3

7. In Figure 3, what is the
   a. perimeter of the right triangle? ($\sqrt{3}+1.5+\sqrt{3}/2$ $=4.098$ units)
   b. area of the right triangle? ($½ \text{x} \sqrt{3}/2\text{x}1.5=6.5$ square units. Also, it is ½ the area of the equilateral triangle.)

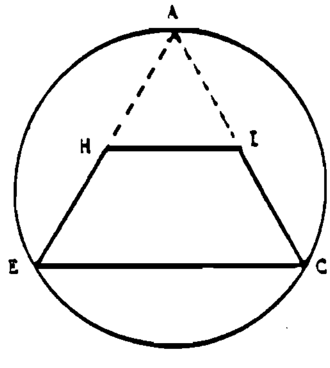


Figure 4

8. In Figure 4, what is the
   a. perimeter of the isosceles trapezoid? ($3\text{x} \sqrt{3}/2 + \sqrt{3}=4.33$ units. The sides and top base have a measure of $\sqrt{3}/2$ and the bottom base a measure of $\sqrt{3}$.)
   b. area of the isosceles trapezoid? (3/4x1.299= 0.974 square units)

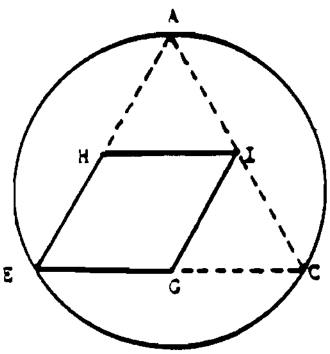


Figure 5

9. In Figure 5, what is the
   a. perimeter of the rhombus? ($4 \sqrt{3}/2=2 \sqrt{3}=3.46$ units)
   b. area of the rhombus? (½x1.299=.649 square units)

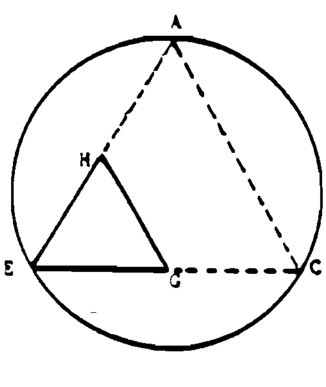


Figure 6

10. In Figure 6, what is the
    a. perimeter of the small equilateral triangle? ($3 \sqrt{3}/2=2.6$ units. Also ½ that of the large equilaterial triangle or 2.6 units.
    b. area of the small equilateral triangle? (1/4 that of the large equilateral triangle or .325 square units)

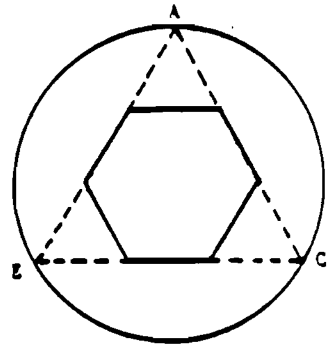


Figure 7

11. In Figure 7, what is the
    a. perimeter of the large hexagon? (Since each side of the original triangle has been trisected the measure of the six sides of the hexagon are equal to the measure of two sides of the triangle or $2 \sqrt{3}=3.46$ units. Note that the original triangle has been divided into nine congruent equilateral triangles. Hence each side of the triangle must have been trisected.)
    b. area of the large hexagon? (6/9 or 2/3 that of the original triangle, or 2/3x1.299=.866 square units)
12. In Figure 7, the large hexagon has been divided into six smaller hexagons, each of which is an equilateral triangle. What is the
    a. perimeter of the small hexagon? (1.73 units or ½ the perimeter of the large hexagon)
    b. area of the small hexagon? (1/6 that of the large hexagon or 1/9 that of the original triangle. 1/6x .866=.144 square units)

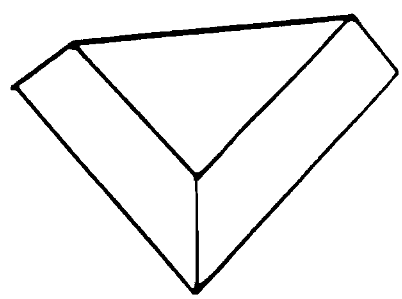


Figure 8

13. In Figures 8, what is the total surface area of the truncated tetrahedron? (From the previous section we know that the area is 7/9 of the area of the large triangle. Therefore, the total surface area is 7/9x1.299=1.01 square units. This is about 32.2% of the area of the circle.)