The Pythagorean Theorem and Area: Postulates into Theorems

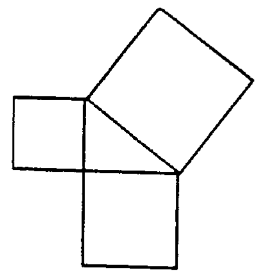
Paul A. Kennedy
Department of Mathematics
Southwest Texas State University
San Marcos, TX 78666-4616
pk03@swt.edu

Contributing Author:
Kenneth Evans
Department of Mathematics, Retired
Southwest Texas State University
San Marcos, TX 78666-4616

Considerable time is spent in high school geometry building an axiomatic system that allows students to understand and prove interesting theorems. In traditional geometry classrooms, the theorems were treated in isolation with some of the more interesting and powerful theorems posed as only postulates. NCTM's *Curriculum and Evaluation Standards* (1989) called for a rethinking of the structure of geometry.

In particular, students should be given an opportunity to discover the ideas of geometry through concrete experiences and direct measurement so that they can build intuition for the central elements in the axiomatic structure. The more recent *Principles and Standards 2000* (NCTM, 2000) calls for a return to reasoning and proof through the K-12 curriculum. The learning of geometry is an inductive/deductive process. Students should experience specific instances that allow them to generalize the postulates, theorems and definitions of geometry. Many of the ideas of geometry can be easily introduced in a discovery setting in which students explore the ideas of measure, congruence, inequality, parallelism and similarity. Once students have inductively acquired an understanding of the ideas of the axiomatic system through these concrete experiences, they can *then* deductively explore short sequences of interesting theorems that demonstrate the elegance of the axiomatic system.

This article deals with the deductive process, highlighting some central theorems in geometry which are too frequently bypassed as postulates in the standard geometry texts. It is curious, for example, that the familiar similar-triangle proof of the Pythagorean theorem is based on something called the Angle-Angle Similarity *Postulate.* When one takes this circuitous route to the Pythagorean Theorem the notion of area never appears. Yet, Euclid's proof depends largely on the notion of area, as shown below. He simply shows that the sum of the area of the two smaller squares is equal to the area of the square on the hypotenuse.



Lightner (1991) speculates on the method of the Pythagoreans when he describes an algebraic/geometric approach that involves dissecting squares and using the idea of combining areas. It seems that area is an essential component in the various proofs of the Pythagorean Theorem.

The following two sequences of theorems include some of the standard "postulates" and culminates with an interesting "area" proof of the Angle-Angle Similarity Theorem which would then allow us to prove the Pythagorean Theorem by the usual similar triangle approach. The theorems are found in a variety of texts, but rarely are they found in high school geometry texts. When the synthetic approach to geometry *is* emphasized, it is important that theorems be arranged in meaningful sequences and that they are connected so that students can understand and connect the various elements of the axiomatic system In much of what follows we use the important idea of one-to-one correspondence given by the Ruler and Protractor Postulates.

TRIANGLE CONGRUENCE

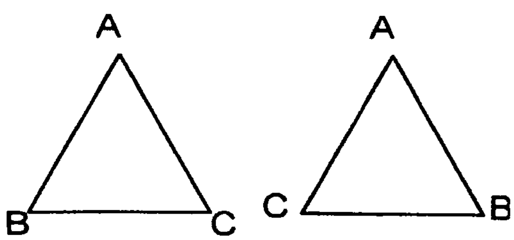
High school geometry texts typically pose SAS, SSS and ASA as postulates. Here we postulate SAS and develop proofs for the other two.

Postulate (SAS): If two sides and the included angle of one triangle are congruent to the corresponding parts of a second triangle, then the triangles are congruent.

The Isosceles Triangle Theorem is an immediate consequence of the SAS Postulate if we take the following transformational perspective. The proof is elegant and simple.

Theorem (Isosceles Triangle): The base angles of an isosceles triangle are congruent.

Proof: Consider isosceles ABC with AB=AC from two perspectives.



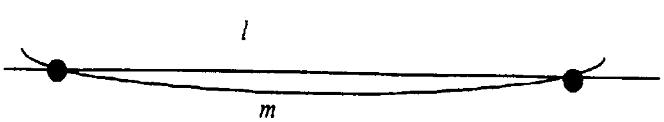
From left to right, since AB=AC and AC=AB with $<A \cong <A$, we have ΔABC ≅ ΔACB by SAS. The corresponding angles B and C are then congruent by definition of congruent triangles.

We are almost ready to investigate the proofs of SSS and ASA. It is important that students spend some time with proof by contradiction. We digress momentarily to develop a simple example using one of the first postulates in the axiomatic system.

Postulate: Two distinct points determine exactly one line.

Theorem: When two distinct lines intersect, they intersect in exactly one point.

Proof: Suppose not. Suppose, given distinct lines *l* and *m*, they intersect in two points.

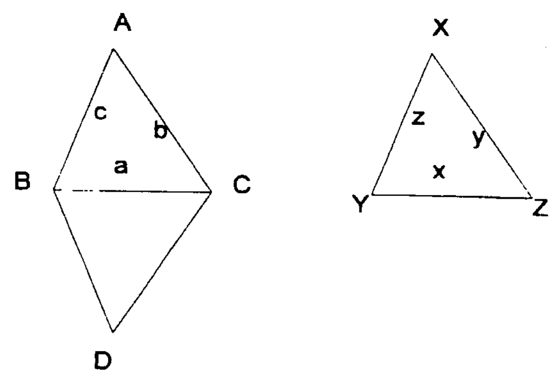


Since two points determine exactly one line, we contradict the hypothesis that *l* and *m* are distinct. Conclude that two distinct lines can intersect in only one point.

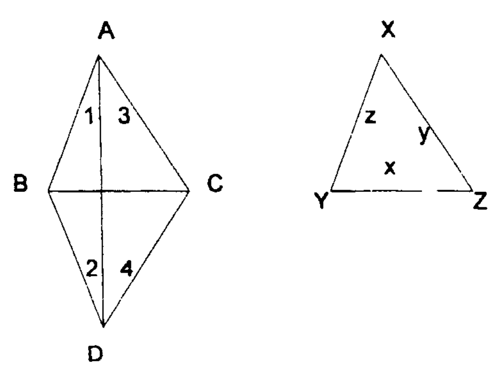
In proof by contradiction, we assume the hypothesis (distinct lines) and the negation of the conclusion (not one point) and reach a contradiction forcing us to accept the conclusion. We use this idea later in the ASA Theorem.

Theorem (SSS): If three sides of one triangle are congruent to the corresponding sides of a second triangle, then the triangles are congruent

Proof: Here we will use the SAS Postulate twice. Consider the two triangles shown with a = x, b = y and c = z. The Protractor and Ruler Postulates allow us to consider <CBD ≅ <Y with BD = z as shown. Now with a=x we have ΔXYZ ≅ ΔDBC by SAS.

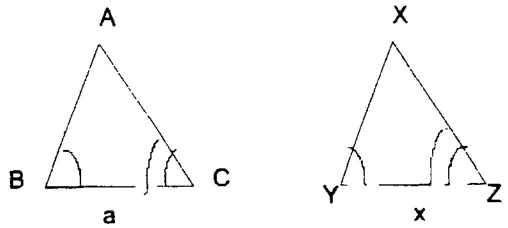


Now we will show that ΔABC ≅ ΔDBC. Consider segment AD. By the Isosceles Triangle Theorem we have <1 ≅ <2 and <3 ≅ <4. Applying the Angle Addition Postulate, we have <BAC ≅ <BDC and ΔABC ≅ ΔDBC by SAS.



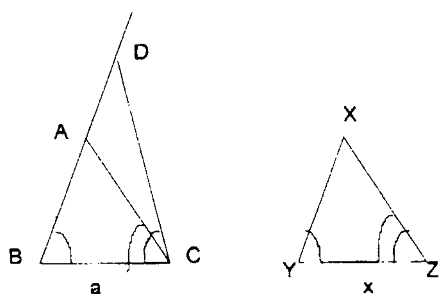
Now ΔABC ≅ ΔXYZ by transitivity as desired.

Theorem (ASA): If two angles and the included side of one triangle are congruent to the corresponding parts of a second triangle, then the triangles are congruent.



Proof: Consider the two triangles shown with a = x, <B = <Y and <C = <Z.

We proceed by supposing the two triangles are not congruent. In particular, suppose AB≠YX. On ray BA consider BD = YX and consider CD.



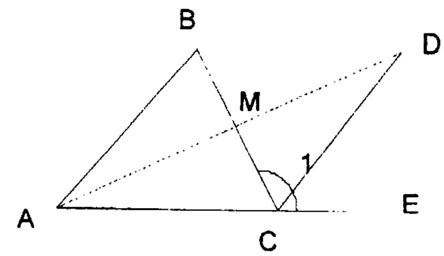
Now we have ΔDBC ≅ ΔXYZ by SAS and corresponding angles BCD and Z congruent. By hypothesis <C ≅ <Z. This contradicts the Protractor Postulate which only allows one ray to determine an angle. Thus ΔABC = ΔXYZ.

GETTING TO THE PYTHAGOREAN THEOREM

The familiar Exterior Angle Equality Theorem, which states that an exterior angle is equal icon measure to the sum of its remote interior angles, is preceded by what we will call the Greater Exterior Angle Theorem, which is often misplaced in the high school texts. In fact, this theorem is usually proved *after* the Exterior Angle Equality theorem because of unnecessary assumptions. Yet, the Greater Exterior Angle Theorem allows us to establish the Equality Theorem and, more importantly, sets the stage for the Pythagorean Theorem proof by similar triangles as evidenced in the following discussion.

Theorem (Greater Exterior Angle): An exterior angle of a triangle is greater in measure than either of its remote interior angles.

Proof: Consider triangle ABC with exterior angle 1. Through the midpoint M of segment BC consider segment AD such that AD is twice AM. Considering the two vertical congruent angles and the bisected segments AD and BC we have ΔABM ≅ΔDCM by SAS.



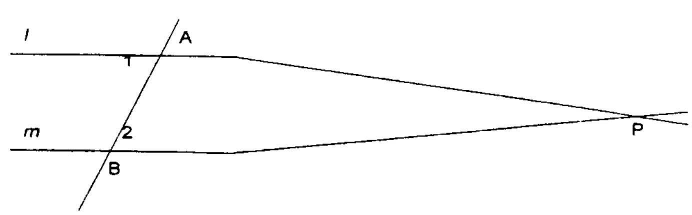
Corresponding angles DCM and B are congruent. Now m<1-m<DCM = m<DCE and since m<DCE>0 we have m<1-m<DCM >0. Then m<1>m<DCM and by substitution m<1 > m<B as desired. A similar construction will show that m<1>m<A.

With the Greater Exterior Angle Theorem behind us, we can now turn the Alternate Interior Angle Postulate into the Alternate Interior Angle Theorem. But, first, we need a very important, controversial postulate.

Postulate (Parallel): In a plane, through a point outside a line, there is exactly one parallel to the line.

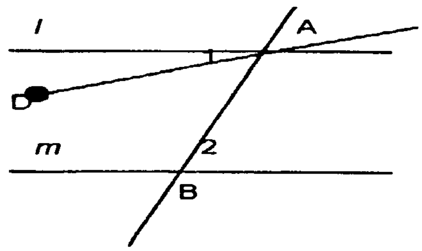
Theorem: Alternate interior angles formed by two lines and a transversal are congruent if and only if the lines are parallel.

Proof: We begin by showing that congruent alternate interior angles imply parallel lines. This part is usually posed as a postulate. Suppose alternate interior angles 1 and 2 are congruent but the lines *l* and *m* are not parallel. Suppose they intersect in some point P as shown.



By the Exterior Angle Theorem, <1 must be larger than <2, but by hypothesis we know that they are congruent. Thus we have reached a contradiction and conclude that *l* and *m* are parallel.

Conversely, suppose we know that *l* and *m* are parallel. Suppose angles 1 and 2 are not congruent. At A consider <BAD ≅ <2 as shown.



The previous result tells us that line AD must be parallel to *m* since the alternate interior angles BAD and 2 are congruent. Now we have two lines parallel to *m* through A. This contradicts the Parallel Postulate, so we conclude that <1 = <2. The following theorems follow immediately from these results and will be of use later.

Theorem: The sum of the measures of the angles of a triangle is 180.

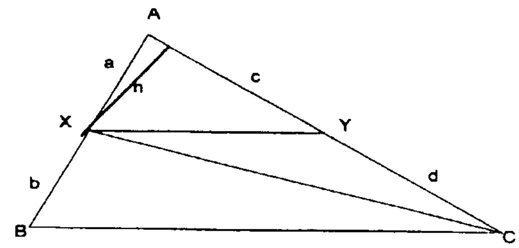
Theorem (Exterior Angle Equality): An exterior angle of a triangle is equal in measure to the sum of its two remote interior angles. (Ironically, in many texts, the more powerful Greater Exterior Angle Theorem follows here.)

Theorem: Corresponding angles formed by two lines and a transversal are congruent if and only if the lines are parallel.

We state the above without proof, so that we can turn our attention to proportionality and similarity, ideas that allow us to connect these theorems and postulates as the foundation for the Pythagorean Theorem. We are almost ready to prove the AA Similarity Theorem. We begin by proving the following important theorem which is often proved *after* the AA Similarity Postulate. It can, however, be proved first using the notion of area and turns out to be a necessary condition for the AA Theorem. It is necessary to assume area of a square and the resultant area of a triangle theorem for the following.

Theorem (Proportional Segments): A line parallel to one side of a triangle that intersects the other two sides in distinct points divides those two sides into proportional segments.

Proof: Consider triangle ABC with XY parallel to BC as shown. We would like to show that a:b=c:d. Consider segments XC and the altitude from X to AY with length *h*. Observe that this is the altitude for both triangles AXY and XCY to bases with lengths *c* and *d* respectively.



Now using α to denote area,

$$\alpha(AXY) = \frac{1}{2}hc$$

$$\alpha(CXY) = \frac{1}{2}hd$$

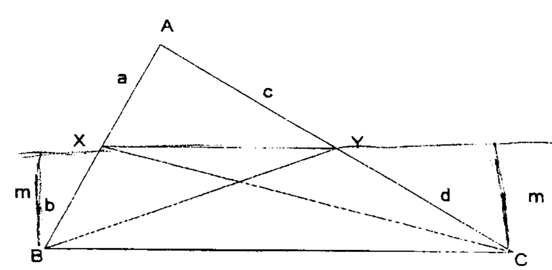
and if we consider the ratio of the areas of the two triangles we have

$$\frac{\alpha(AXY)}{\alpha(CXY)} = \frac{\frac{1}{2}hc}{\frac{1}{2}hd} = \frac{c}{d}.$$

Similarly by considering the segment from B to Y and the altitude with length *k* from Y to AX we can show:

$$\frac{\alpha(AXY)}{\alpha(BXY)} = \frac{\frac{1}{2}ka}{\frac{1}{2}kb} = \frac{a}{b}.$$

Now, if we can show the areas of CXY and BXY equal, we are done. Since parallel lines can easily be shown to be equidistant, the altitude of both triangles to the common base, segment XY, have the same length, *m*, as shown below.



Thus both triangles have area $\frac{1}{2}m \bullet XY$.

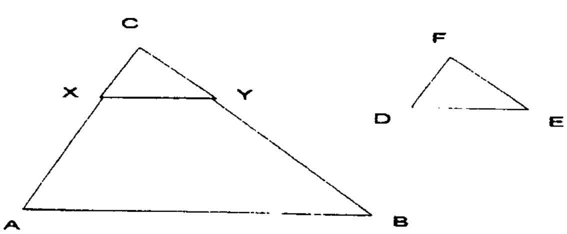
And then

$$\frac{a}{b}=\frac{\alpha(AXY)}{\alpha(BXY)}=\frac{\alpha(AXY)}{\alpha(CXY)}=\frac{c}{d}.$$

We are now ready to prove the Angle-Angle (AA) similarity theorem. It is easy to show that if two angles of one triangle are congruent to two angles of a second triangle, then the third angles from each are likewise congruent. We will need this in the proof.

Theorem (AA Similarity): If two angles of one triangle are congruent to two angles of a second triangle, then the triangles are similar.

Proof: In order to establish similarity, we must show that the sides are proportional. Consider triangles ABC and DEF with <A = <D and <C = <F. On segments CA and CB locate points X and Y such that CX=FD and CY = FE.



Now ΔDEF = ΔXYC by SAS and <A = < CXY by transitivity imply that XY | | AB. Applying the Proportional Segments Theorem, we just proved we know that

$$\frac{CX}{XA}=\frac{CY}{YB}$$

Using the definition of between and the fact that if a:b =c:d then a:a+b = c:c+d, we have

$$\frac{CX}{CA}=\frac{CY}{CB}$$

and finally by substitution

$$\frac{FD}{CA}=\frac{FE}{CB}.$$

By a similar method we can show the other sides proportional.

It is important for students to see that the idea of area can be used to prove the AA Similarity Theorem which, in turn, allows for the similar triangle proof of the Pythagorean Theorem. In fact, many of the so-called "postulates" in the secondary texts can be proved as theorems without much difficulty. When AA similarity is postulated we lose sight of the importance of our axiomatic system. Students can see how the SAS postulate, the Greater Exterior Angle Theorem and the Parallel Postulate combine forces to lay the groundwork for perhaps the most important theorem in Euclidean Geometry. One might be led to believe that the Pythagorean Theorem cannot be proved directly without area. Interestingly, Moise (1990) provides an elegant proof of the AA Similarity Theorem without area. While the proof may be beyond the scope of high school geometry, it is worth noting that one can, in fact, arrive at the Pythagorean Theorem without area.

Much of the richness of geometry is lost when theorems are treated in isolation and when key theorems are bypassed as postulates. Part of "problem posing" in geometry should include an investigation of the way the axiomatic system fits together. How do some key theorems like the Greater Exterior Angle Theorem allow us to generate important geometric ideas? For what later theorems is the Parallel Postulate a necessary condition? NCTM's *Principles and Standards 2000* challenges us to reevaluate both the content and methodology of geometry instruction. When the synthetic or analytic approach is taken, posing short sequences of interesting connected theorems encourages problem solving that engenders deeper understanding and appreciation of geometry.

REFERENCES

Lightner, James E. "A Chain of Influence in the Development of Geometry." *Mathematics Teacher* 84 (Jan 1991):15-19.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA. National Council of Teachers of Mathematics, 1989.

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 2000.

Moise, Edwin E. Elementary Geometry from an Advanced Standpoint. Reading Massachusetts: Addison Wesley Publishing Company, 1990.