Patterns in Figurate Sequences

Concepts

- Numerical patterns
- Figurate numbers: triangular, square, pentagonal, hexagonal, heptagonal, octagonal, etc.
- Closed form representation of a number sequence
- Function notation and graphing
- Discrete and continuous data

Materials

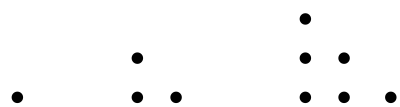
- Chips, two-color counters, or other manipulatives for modeling patterns
- Student activity sheet "Patterns in Figurate Sequences"
- TI-73 EXPLORER or TI-83 Plus/SE

Introduction

Mathematics has been described as the "science of patterns." Patterns are everywhere and may appear as geometric patterns or numeric patterns or both. Figurate numbers are examples of patterns that are both geometric and numeric since they relate geometric shapes of polygons to numerical patterns. In this activity you will analyze, extend, and describe patterns involving figurate numbers and make connections between numeric and geometric representations of patterns.

Student Activity Sheet

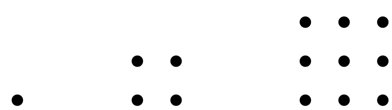
1. Using chips or other manipulatives, analyze the following pattern and extend the pattern pictorially for two more terms.



2. Write the sequence of numbers that describes the quantity of dots above.

3. Describe this pattern in another way.

4. Extend and describe the following pattern with pictures, words, and numbers.



5. Analyze Table 1. Fill in each of the rows of the table.

Table 1: Figurate Numbers

Figurate Number	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	***n*th**
Triangular	1	3	6	10	15	21	28	36	$n(n+1)/2$
Square	1	4	9	16	25	36	49	64	
Pentagonal	1	5	12	22	35	51	70		
Hexagonal	1	6	15	28	45	66			
Heptagonal	1	7	18	34	55				
Octagonal	1	8	21	40					
Nonagonal	1	9	24						
Decagonal	1	10							
Undecagonal	1								
...									
N-gonal	1	n							

6. What patterns do you observe? List all of the patterns you can find in this table.

7. What does it mean to say the *n*th term?

8. Following the instructions below, use the TI-73 EXPLORER or TI-83 Plus/SE to complete Table 2.
   - Fill in the **Figurate Number**, **Numbers in the Set**, and the ***n*th Term** columns from Table 1.
   - Write the closed form representation for each figurate number.
   - Write each of the sequences using function notation, e.g., $T(n) = n(n+1)/2$ or $f(x) = x(x+1)/2$.
   - Graph the functional representation for each figurate number using a TI-73 Explorer or TI-83 Plus/SE and sketch the graph in your table.

Table 2: Connecting Figurate Numbers To Functional Notation

Figurate Number	Numbers in the Set	n^{th} Term	Closed Form Representation	Functional Notation	Graph

9. Explain whether or not the graphs represent the figurate numbers given in Table 1 and the "numbers in the set" given in Table 2. Consider which points on the graphs actually represent figurate numbers and what the other points represent. (Hint: What values of *x* give you the values in Table 1?)

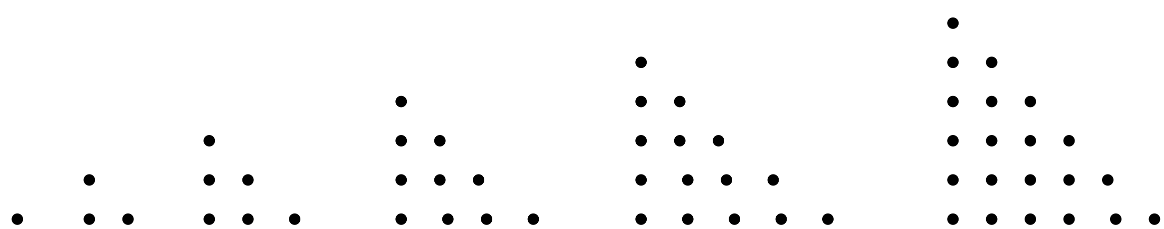
10. What is the difference between discrete data and continuous data? How does this relate to the graphs you have completed?

Teacher Notes

Introduction
Mathematics has been described as the "science of patterns." Patterns are everywhere and may appear as geometric patterns or numeric patterns or both. Figurate numbers are examples of patterns that are both geometric and numeric since they relate geometric shapes of polygons and to numerical patterns. This activity is designed to help elementary, middle grades, and secondary students analyze, extend, and describe patterns and to make connections between numeric and geometric representations of patterns.

At the elementary and middle grades levels, children should first use chips or other manipulatives to build models for triangular, square, and pentagonal numbers. They should extend the patterns concretely and pictorially, and then describe them numerically. For young children, you may choose to omit having them find the *n*th term representation in Table 1 and use only questions #1-6. The TI-73 EXPLORER or TI-83 Plus/SE is then used to graph each closed form and functional representations.

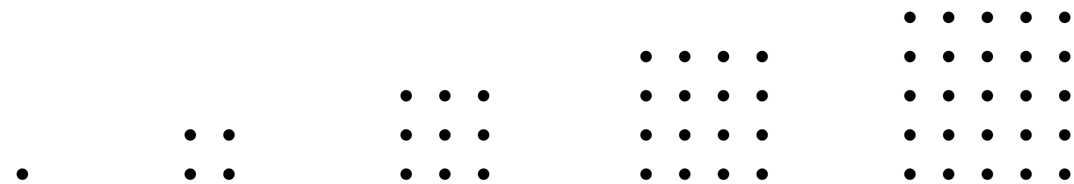
Instructions
1. Distribute the "Patterns in Figurate Sequences" student activity sheet pages, manipulatives (chips, two-color counters, etc.) for modeling patterns, and the TI-73 EXPLORER or TI-83 Plus/SE. Have students use the manipulatives to extend the first pattern and record their pictorial solution. Engage them in a discussion of their answers.



2. Depending on the level and experience of the students, you may need to explain how to write a sequence of numbers before asking them to write the sequence of numbers that describes their completed dot pattern.
   Answer: *{1, 3, 6, 10, 15, 21,…}*

3. Have students record how their completed dot pattern could be described in another way. Engage them in a discussion of their answers. If the term "triangular" has not yet arisen in the discussion, introduce it as well as the term "figurate number" before continuing.
   Answer: *It can be described in terms of the whole number representing the chips or manipulatives (1,3,6,10, 15, 21, …) or it can be described by rows of the triangle (1 x 1 = 1, 2 x 1 + 1 x 1 = 3, 3 x 1 + 2 x 1 + 1 x 1 = 6, …).*
   Can it be described numerically in other ways?

4. Ask students to extend and describe the next "square" pattern with pictures, words, and numbers. Engage them in a discussion of their answers. Make certain to reinforce the "figurate number" term.



Answer: {1, 4, 9, 16, …} OR 1 x 1, 2 x 2, 3 x 3, 4 x 4, … OR 1+0, 2+2, 3+3+3, 4+4+4+4, … OR $1^2, 2^2, 3^2, 4^2$, …

5. Have students complete and analyze **Table 1: Figurate Numbers**. The TI-73 EXPLORER or TI-83 Plus/SE may be helpful to the students in completing the table.

Table 1: Figurate Numbers

Figurate Number	**1st**	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	**nth**
Triangular	1	3	6	10	15	21	28	36	$\frac{n(n+1)}{2}$
Square	1	4	9	16	25	36	49	64	n^2
Pentagonal	1	5	12	22	35	51	70	100	$\frac{n(3n-1)}{2}$
Hexagonal	1	6	15	28	45	66	91	136	$\frac{n(4n-2)}{2}$
Heptagonal	1	7	18	34	55	81	112	172	$\frac{n(5n-3)}{2}$
Octagonal	1	8	21	40	65	96	133	208	$\frac{n(6n-4)}{2}$
Nonagonal	1	9	24	46	75	111	154	244	$\frac{n(7n-5)}{2}$
Decagonal	1	10	27	52	85	126	176	280	$\frac{n(8n-6)}{2}$
Undecagonal	1	11	30	58	95	141	197	316	$\frac{n(9n-7)}{2}$
…									
N-gonal	1	n							

6. Ask students to describe and list all of the patterns they can find in Table 1. Engage students in a discussion of their answers.

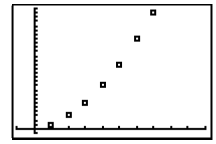
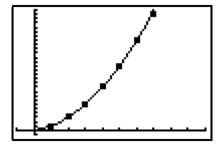
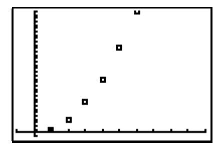
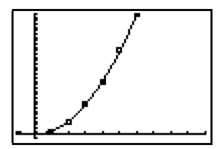
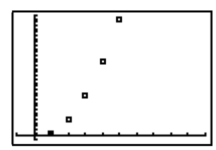
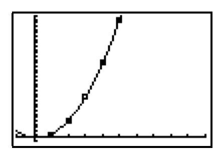
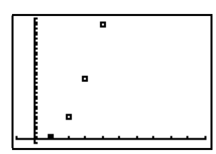
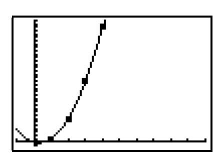
   Answer: *Many different answers will be noted such as: (1) All entries in the first column are 1's; (2) The second entry has the number of sides of the polygon by the same name, except "square" numbers. A square is a quadrilateral, however. (3) The patterns of differences in the rows and columns are a pattern. <u>Columns:</u> The same finite difference is noted in each vertical column, depending upon which column is chosen. Example: The 1st column has 0 as the difference between each term, the 2nd column has a difference of one between each vertical term, the 3rd column has a finite difference of 3 between each term in the column, the 4th column has a finite difference of six between each term in the column, etc. (4) <u>Rows:</u> There is a pattern of differences in each row of the sequences. Example: Between the first and second terms of the triangular numbers the difference is 2; the difference between the second and third terms is 3; the difference between the third and 4th term is four, etc.; however, the difference between the 1st and 2nd square numbers is 3; the difference between the 2nd and 3rd square numbers is 5; the difference between the 3rd and 4th square numbers is 7; etc. Are there other patterns that are observed? What about the pattern of the nth term?*

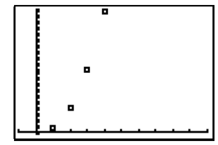
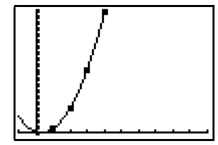
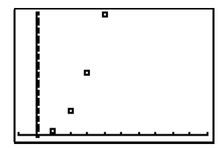
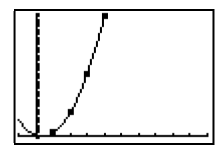
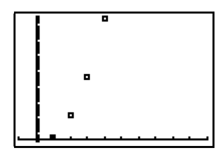
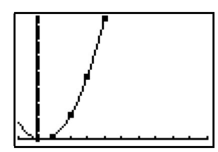
7. Lead a discussion of what it means to say the "nth term". Make certain that students connect this term with the closed form representation of a sequence. Then have students record their description/definition on their student activity sheet.
   Answer: *The nth term is the general way of describing the sequence of numbers. As n takes on values of 1, 2, 3, …, a sequence of numerical values is generated. Each value corresponds to the term specified by the value of n and is a term of the sequence. This general description of the term leads to the closed form, or an expression representing each figurate number. Example: $T = n(n+1)/2$ where T represents any triangular number. If one wants the 4th triangular number, $n = 4$ and $T_4 = 4(5)/2$ or $T_4 = 10$. If one wants the 50th triangular number, then, $T_{50}= 50(51)/2$ or $T_{50} = 1275$.*

8. Explain the instructions below and have students use the TI-73 EXPLORER or TI-83 Plus/SE to complete **Table 2: Connecting Figurate Numbers to Functional Notation**. A review of function notation and graphing functions using the TI-73 EXPLORER or 83 Plus/SE may be necessary.
   - Fill in the **Figurate Number**, **Numbers in the Set**, and the ***n*th Term** columns from Table 1.
   - Write the closed form representation for each figurate number.
   - Write each of the sequences using function notation, e.g., $T(n) = n(n+1)/2$ or $f(x) = x(x+1)/2$.
   - Graph the functional representation for each figurate number using a TI-73 EXPLORER or TI-83 Plus/SE and sketch the graph in your table.

   Answer: *Although this question makes a "leap" in mathematical understanding for some students, others will find a very interesting connection between the patterns observed in the figurate numbers, the geometric representation of those numbers, and the nth term notation for each set of figurate numbers. Extending to the notation of functions makes an even nicer "jump" and connection for students. A summary of this information is shown in Table 2. NOTE: A scatterplot of the figurate numbers is included on the answer key with graph of the function.*

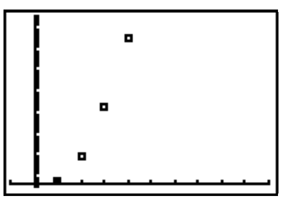
Table 2: Connecting Figurate Numbers To Functional Notation

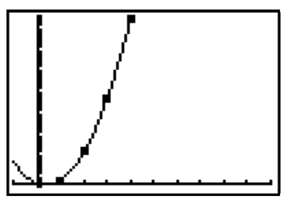
Figurate Number	Numbers in the Set	n^{th} Term	Closed Form Representation	Functional Notation	Graph
Triangular	{1, 3, 6, 10,…}	$n(n+1)/2$	$T_n = n(n+1)/2$ or $T_n = (n^2 + n)/2$	$T(n) = n(n+1)/2$ or $T(n) = (n^2 + n)/2$ or $f(x) = x(x+1)/2$	 
Square	{1, 4, 9, 16, …}	n^2	$S_n = n^2$	$S(n)= n^2$ or $f(x) = x^2$	 
Pentagonal	{1, 5, 12, 22, 35, ...}	$n(3n-1)/2$	$P_n = n(3n-1)/2$ or $P_n = (3n^2-n)/2$	$P(n) = n(3n-1)/2$ or $P(n) = (3n^2-n)/2$ or $f(x) = x(3x-1)/2$ or $f(x) = (3x^2-x)/2$	 
Heptagonal	{1, 7, 18, 34, 55, 81,…}	$n(5n-3)/2$	$HP_n = n(5n-3)/2$ or $HP_n = (5n^2-3n)/2$	$HP(n) = n(5n-3)/2$ or $HP(n) = (5n^2-3n)/2$ or $f(x) = x(5x-3)/2$ or $f(x) = (5x^2-3x)/2$	 

Octagonal	{1, 8, 21, 40, 65,…}	$n(6n-4)/2$	$O_n = n(6n-4)/2$ or $O_n = (6n^2-4n)/2$ or $O_n = 3n^2-2n$	$O(n) = n(6n-4)/2$ or $O(n) = (6n^2-4n)/2$ or $O(n) = 3n^2-2n$ or $f(x) = x(6x-4)/2$ or $f(x) = 3x^2-2x$	 
Nonagonal	{1, 9, 24, 46, 75…}	$n(7n-5)/2$	$N_n = n(7n-5)/2$ or $N_n = (7n^2-5n)/2$	$N(n) = n(7n-5)/2$ or $N(n) = (7n^2-5n)/2$ or $f(x) = x(7x-5)/2$ or $f(x) = (7x^2-5x)/2$	 
Decagonal	{1, 10, 27, 52, 85, …}	$n(8n-6)/2$	$D_n = n(8n-6)/2$ or $D_n = (8n^2-6n)/2$ or $D_n = 4n^2-3n$	$D(n) = n(8n-6)/2$ or $D(n) = (8n^2-6n)/2$ or $D(n) = 4n^2-3n$ or $f(x) = x(8x-6)/2$ or $f(x) = 4x^2-3x$	 

Comments: The figurate numbers are shown in the first graph of each specified row. Only the points denoted by squares actually represent figurate numbers. The second graph in each row is the closed form/functional representation and includes points that are not in the set of figurate numbers. The domain for figurate numbers is the set of positive integers.

9. Have students explain in writing on the student activity sheet whether or not the graphs represent the figurate numbers given in Table 1 and the "numbers in the set" given in Table 2. (Hint: What values of x give you the values in Table 1?)
   Engage students in a discussion of which points on the graphs actually represent figurate numbers and why the other points are not in the set of figurate numbers. The use of guiding questions will facilitate students' discovery that the domain for figurate numbers is the set of positive integers.
10. Have students explain in writing the difference between discrete data and continuous data and relate their response to the graphs they completed. As they attempt to relate this to the graphs they have completed, use the TI-73 EXPLORER or TI-83 Plus/SE to graph a scatter plot and overlay the corresponding function. Using a "square" as the mark on the scatter plot makes the relationship more obvious.
    Answer: *Continuous data is data that includes every of a set; however, discrete data has some of the members missing. Consider the set of real numbers, which can be represented by the points on a graph. If we consider the decagonal numbers {1, 10, 27, …) on the real plane, we notice that many points are not included between each of the numbers. The second graph represents all the numbers in the function, D(n) = n(8n-6)/2. The graph is continuous – from one point to the next one. See the graphs in Table 2 and the two below.*





And the numbers go on and on….