

Module 9 - The Relationship between a Function and Its First and Second Derivative

Introduction

In previous modules you used the graph of a function to investigate its derivative. In this module you will use the derivative to find properties of the original function. You will also investigate what the second derivative says about a function.

Lesson Index:

9.1 - What the First Derivative Says About the Function

9.2 - What the Second Derivative Says About the Function

After completing this module, you should be able to do the following:

- Use the Derivative and Tangent features in the Graph Math menu
- Describe what the first derivative says about a function
- Use the *tanimate* program to illustrate the relationships between a function and its first and second derivatives
- Use the Inflection feature in the Graph Math menu
- Use the Derivative key to find second derivatives
- Describe what the second derivative says about a function

Lesson 9.1: What the First Derivative Says About the Function

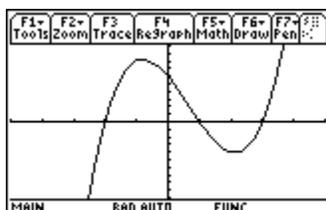
In Module 8 we saw that the value of the derivative of f at x is given by the slope of the line tangent to the graph of f at x . In this lesson you will explore what the first derivative says about the graph of the original function by using the Derivative and Tangent features of the calculator.

If f is a function, then its first derivative is denoted by f' , which is read "f prime," and the value of the first derivative at $x = a$ is $f'(a)$. The calculator denotes the first derivative of f as $\frac{d}{dx}(f(x))$, which is an alternate notation. If $y = f(x)$, other notations include $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$, and dy/dx .

Using the Derivative Feature on the Graph Screen

Follow the procedure given below to graph a function and use the Derivative feature of the Graph screen's Math menu to compute its derivative.

- Graph $y = x^3 - 2x^2 - 5x + 6$ in a $[-5, 5] \times [-10, 10]$ window



Find the value of the derivative at $x = -2$ with the Derivative feature in the F5:Math menu.

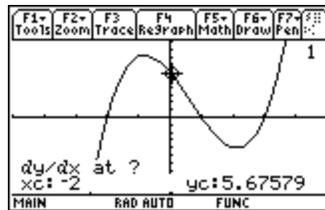
- Display the Math menu by pressing **F5**
- Highlight 6:Derivatives by using the cursor movement keys
- Open the Derivative submenu by pressing **⏏**



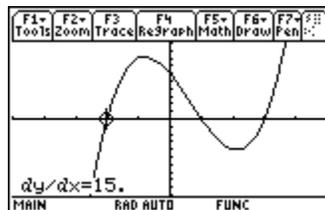
- Select dy/dx by pressing **ENTER**

You should see the graph and a prompt for the x -coordinate at which the derivative is to be evaluated.

- Enter -2 as the x -coordinate by pressing **(-)****2**



- Set the value of the x-coordinate by pressing **ENTER**

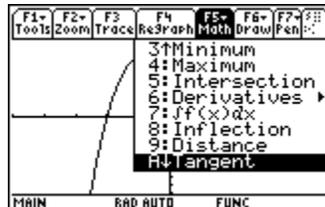


The value of the derivative of $f(x) = x^3 - 2x^2 - 5x + 6$ at $x = -2$ is 15.

Using the Tangent Feature

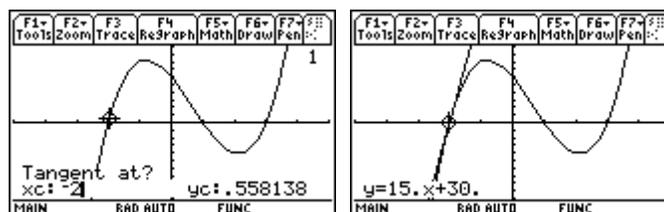
You can use the Tangent feature of the Graph Math menu to see the graph of a tangent line at a specific point along with the equation for that tangent line.

- Open the Math menu and select A:Tangent



After you make the selection, you should see the graph of the function and a prompt for the x-coordinate of the point of tangency.

- Enter -2. Then press **ENTER**



The TI-89 displays the graph of the tangent line and its equation is shown at the bottom of the screen. The equation of the tangent line in the example above is $y = 15x + 30$.

9.1.1 How does the equation of the tangent line display the value of the derivative at $x = -2$?

9.1.2 The table below contains points on the graph of the function $f(x) = x^3 - 2x^2 - 5x + 6$. Use the Derivative feature to find the derivative at each point shown and record the values.

x	-2	-1	0	1	2	3
$f(x)$	0	8	6	0	-4	0
$f'(x)$						

9.1.3 What do you notice about the function values on the intervals where the derivative is positive compared with the function values on the intervals where the derivative is negative?

- Redraw the graph and clear the tangent line by pressing **F4**

Tech Tip!

Derivative and Tangent Features on the Graph Screen

The Derivative and the Tangent features on the graph screen give approximate results. The Derivative command on the Home Screen gives exact results.

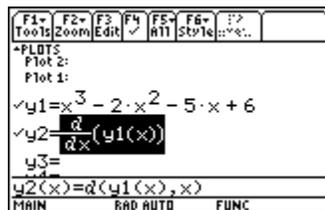
Finding Turning Points from the Derivative

Definition

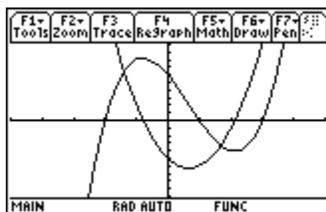
A turning point of the graph of a function is a point where the function changes from increasing to decreasing or changes from decreasing to increasing.

Every [turning point](#) of a function has a derivative of zero at that point if it is differentiable at that point. Graphing the derivative with the function can illustrate how to find these turning points.

- Set $y2=d(y1(x),x)$

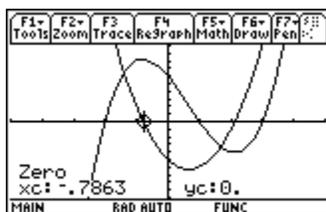


- Display the graph of the function and its derivative by pressing \blacklozenge [GRAPH]



The function is increasing exactly where the derivative is positive, and decreasing exactly where the derivative is negative. On the graph of the derivative find the x-value of the zero to the left of the origin.

- Select the Zero feature in the F5:Math menu
- Select the graph of the derivative by pressing \ominus
A small 2 should appear in the upper right-hand corner to indicate that the graph defined in y2 is selected.
- Set the lower bound by moving the cursor to the left of the desired zero and pressing $\boxed{\text{ENTER}}$
- Set the upper bound by moving the cursor to the right of the desired zero and pressing $\boxed{\text{ENTER}}$



One of the zeros of the derivative is approximately $x = -0.7863$.

9.1.4 Use the Zero feature of the F5:Math menu to find the other zero of the derivative.

Summarizing the Relationship between f and f'

The following characteristics of the function $f(x) = x^3 - 2x^2 - 5x + 6$ can be determined from the graph of its first derivative.

The derivative is zero at $x = -0.7863$ and at $x = 2.11963$, which are the x-values of the turning points.

The function is increasing on the intervals $(-\infty, -0.7863)$ and $(2.11963, \infty)$ because its first derivative is positive there.

The function is decreasing on the interval $(-0.7863, 2.11963)$ because its first derivative is negative there.

Lesson 9.2: What the Second Derivative Says About the Function

In this lesson you will use the [tanimate](#) program and the Inflection feature in the Graph Math menu to determine characteristics of a function by looking at the graph of its second derivative.

Defining the Second Derivative

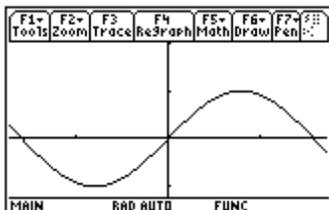
If f is a function, then we have seen that its derivative, f' , is also a function. The function f' may have its own derivative. The derivative of f' is called the second derivative of f .

The second derivative of the function f is denoted by f'' , which is read " f double prime." The calculator denotes a second derivative as $\frac{d^2}{dx^2}(f(x))$, which is an alternate notation.

Visualizing How the Tangent Line Slopes Change

To visualize the relationship between a function and its second derivative, graph a function, run [tanimate](#), and watch the creation of tangent lines with a new focus.

- Graph $y_1 = \sin 2x$ in a $[-1.7, 1.7] \times [-1.2, 2]$ window. Be sure that radian mode is selected.



Notice that the tangent lines have positive slopes on the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \approx (-0.79, 0.79)$, but there is another feature of the tangent lines on this interval that is also important. It concerns how the slopes of the tangent lines change from one point to the next, which the program *tanimate* can illustrate. The final graph produced by *tanimate* does not illustrate this concept, but watching the dynamic process of the animated tangent lines may.

- Run the TI-89 program *tanimate*
- Select "Animated" in the first dialog box
- Select "MEDIUM" in the second dialog box
- Select "TANGENT&PTS" in the third dialog box
- Enter $-\pi/4$ for the left endpoint and $\pi/4$ for the right endpoint

While the program is running watch the tangent lines carefully and pay attention to how the tangent line slopes are changing from one point to the next.

9.2.1 Describe how the slopes of the tangent lines change.

Defining Concavity

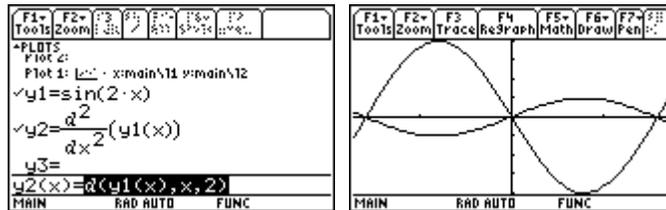
The graph of a function is said to be concave upward on an interval where the slopes of the tangent lines increase and concave downward where the slopes of the tangent lines decrease. That is, concavity is a measure of whether a curve is turning upward or downward.

Determining the Concavity of f from the Graph of f''

The graph of the second derivative of $y = \sin 2x$ can be found by using the $[d]$ key.

- Graph $y_2 = d^2(y_1(x), x, 2)$ in a $[-1.7, 1.7] \times [-4, 4]$ window

Including the "2" as a third argument in the derivative command indicates the second derivative.



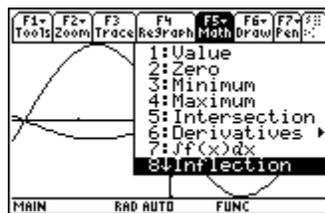
9.2.2 What is the relationship between the second derivative and the original function?

Approximate where the graph of $y = \sin 2x$ shown above changes from concave upward to concave downward?

Finding an Inflection Point

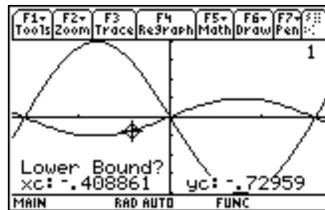
A point where a function's graph changes concavity is called an inflection point. You can find this point with the Inflection feature in the Graph Math menu.

- Open the Math menu by pressing $[F5]$
- Select 8:Inflection

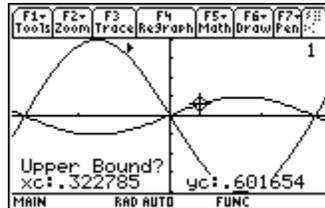


Make sure you are on the right graph. There should be a "1" in the upper right corner of the screen. The calculator prompts you for a lower bound.

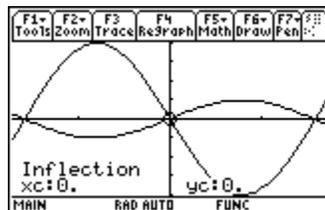
- Move the cursor until it is slightly to the left of the estimated inflection point



- Select the desired point by pressing **ENTER**
- Move the cursor until it is slightly to the right of the estimated inflection point



- Select the desired point by pressing **ENTER**



The function $y = \sin 2x$ has an inflection point at $(0,0)$.

Finding the Inflection Point of Another Function

Find the inflection point of the function given in Lesson 9.1: $y = x^3 - 2x^2 - 5x + 6$.

- Graph $y = x^3 - 2x^2 - 5x + 6$ in a $[-5, 5] \times [-10, 10]$ window
- Find the inflection point by using the Inflection feature

9.2.3 What are the coordinates of the inflection point?

Self Test

Define $f(x) = -x^3 + 2x^2 + 4x - 3$ and graph it in a $[-7.9, 7.9] \times [-6, 6]$ window.

1. Use the Derivative feature of the Graph Math menu to find $f'(1.5)$.
2. Use the Tangent feature of the Graph Math menu to find the equation of the tangent line to $y = f(x)$ at $x = 1.5$.
3. Refresh the graph screen by pressing $\boxed{F4}$ and use the Inflection feature of the Graph Math menu to find the inflection point of $y = f(x)$.
4. Find the intervals where the graph of f is concave upward and where it is concave downward.
5. Display the graphs of the function f and its derivative and use the graph of the derivative to determine where the turning points of f occur.
6. Use the graph of the derivative of f to determine where f is increasing and where f is decreasing.

7. Use the Derivative key to find the second derivative of $y = \frac{x+1}{x-2}$