Developing Geometric Properties through the use of Rigid Motions

Teacher Materials

Colorado State High School Mathematics Standards:

4.1 – Objects in the plane can be transformed, and those transformations can be described and analyzed mathematically.

Developed by: *Rebecca Cooper, Charles Grossen, and Sarah Nelson*

Colorado State University

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Geometry Unit:
Developing Geometric Properties Through the Use of Rigid Motions

Table of Contents

Applicable Colorado High School Standards

Prepared Graduates:

- Apply transformation to numbers, shapes, functional representations, and data

Grade Level Expectation: High School

Concepts and skills students master:

1. Objects in the plane can be transformed, and those transformations can be described and analyzed mathematically

Evidence Outcomes

Students can:

a. Experiment with transformations in the plane.
   - i. State precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
   - ii. Represent transformations in the plane using[i] appropriate tools.
   - iii. Describe transformations as functions that take points in the plane as inputs and give other points as outputs.
   - iv. Compare transformations that preserve distance and angle to those that do not.
   - v. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
   - vi. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
   - vii. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using appropriate tools.
   - viii. Specify a sequence of transformations that will carry a given figure onto another.

b. Understand congruence in terms of rigid motions.
   - i. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure.
   - ii. Given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
   - iii. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
   - iv. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

c. Prove geometric theorems.
   - i. Prove theorems about lines and angles.
   - ii. Prove theorems about triangles.
   - iii. Prove theorems about parallelograms.

d. Make geometric constructions.
   - i. Make formal geometric constructions[ii] with a variety of tools and methods.
   - ii. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

21st Century Skills and Readiness Competencies

Inquiry Questions:

1. What happens to the coordinates of the vertices of shapes when different transformations are applied in the plane?
2. How would the idea of congruency be used outside of mathematics?
3. What does it mean for two things to be the same? Are there different degrees of "sameness?"
4. What makes a good definition of a shape?

Relevance and Application:

1. Comprehension of transformations aids with innovation and creation in the areas of computer graphics and animation.

Nature of Mathematics:

1. Geometry involves the investigation of invariants. Geometers examine how some things stay the same while other parts change to analyze situations and solve problems.
2. Mathematicians construct viable arguments and critique the reasoning of others.
3. Mathematicians attend to precision.
4. Mathematicians look for and make use of structure.

Lesson 1:

Introduction to Rigid Motions

Objective: Students will generalize properties of reflections, rotations, and translations using coordinates.

Assessment: Quiz on definitions of, analyzing, and performing reflections, rotations, and translations.

Colorado State Standards: 4.1 a. ii, iii, vi, vii, b. i

Contents:

- **Notes 1.1**
- **Quiz 1.1**

Notes 1.1

When teaching these definitions open geogebra or draw on board to show students each of these terms as opposed to just reading the definitions.

Line – straight line, infinite in both directions, only one line going through two distinct points. Through point P passes only one line which is parallel to another line which does not contain P.

Parallel – Two lines that do not intersect.

Line segment – the part of the line joining A and B, A & B are endpoints.

Ray – Union of either half line with point P.

Triangle Inequality – Any three points A, B, and C satisfy the inequality then, dist(A,C) <= dist(A,B) + dist(B,C)

Circle – the set of all points in a plane that are equidistant form a given point, the distance is called the radius.

Congruence – A transformation of the plane when it is a composition of a finite number of basic isometries.

There are three isometries of the plane, rotations, reflections, and translations. These three isometries are translations, which are also distance-preserving in the plane.

Add in physical notes for each of these isometries, for instance a cardboard triangle to manipulate.

When given a cut out triangle and grid paper, have students perform a reflection across a predetermined line from a predetermined starting point. Do activities like this for each of these isometries.

ROTATIONS

How to rotate?

To find point of rotation connect original point to the new image of that point. Then find the perpendicular bisector of that segment. Do this for one more point. Where the perpendicular bisectors intersect is the point of rotation.

Set a point of rotation, O. For any point P that you want to rotate, create a circle with center O and radius OP. Then, rotate the point P along the circle by the specified angle, as you would normally draw an angle.

REFLECTIONS

How to reflect?

Have students connect point and image and see that the line of reflection bisects these segments.

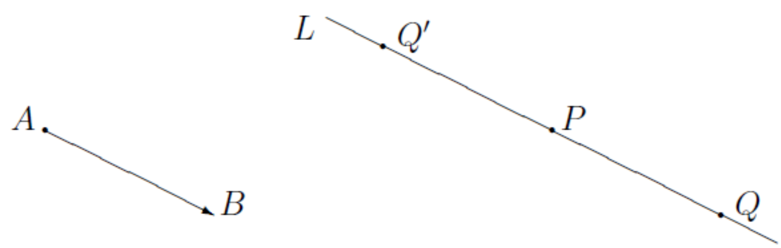
If P is included in L, then R(P) = P. If P is not included in L, then R(P) is the point Q so that L is the perpendicular bisector of the segment PQ.

One reflection changes orientation.

TRANSLATIONS

How to translate?

Using a vector and a distance, draw a line through the point you wish to translate which is parallel to the vector. Then, by measuring off the distance desired, move the point along the line to the new position.



Quiz 1.1

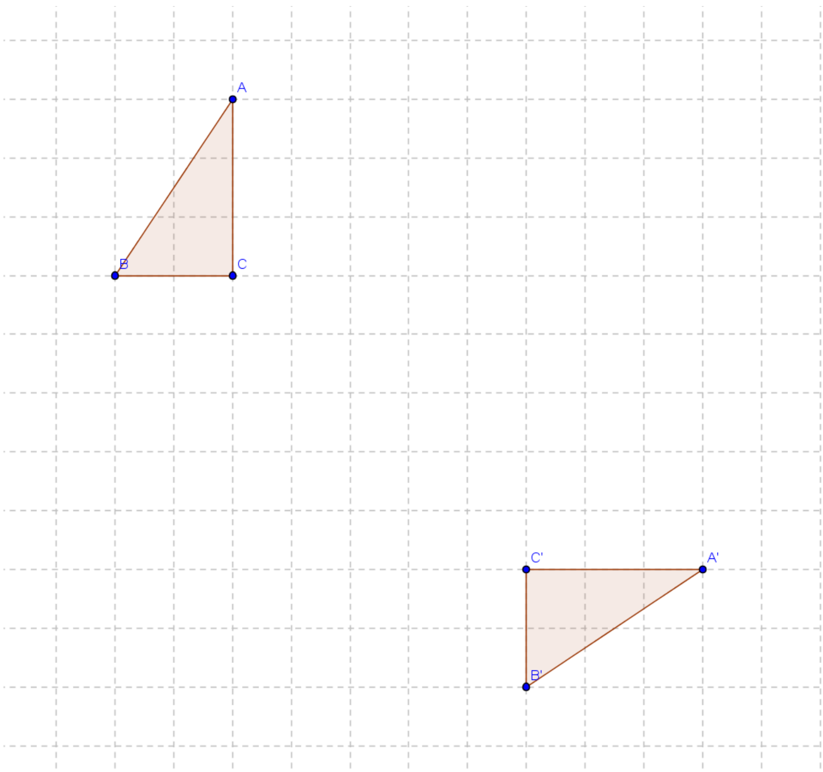
Give the definitions/state how to use each of the following:

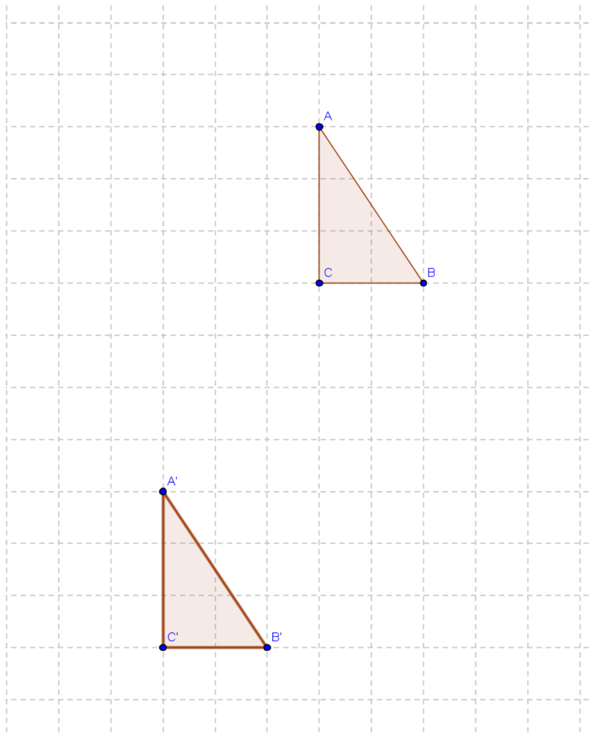
<u>Reflection</u> –

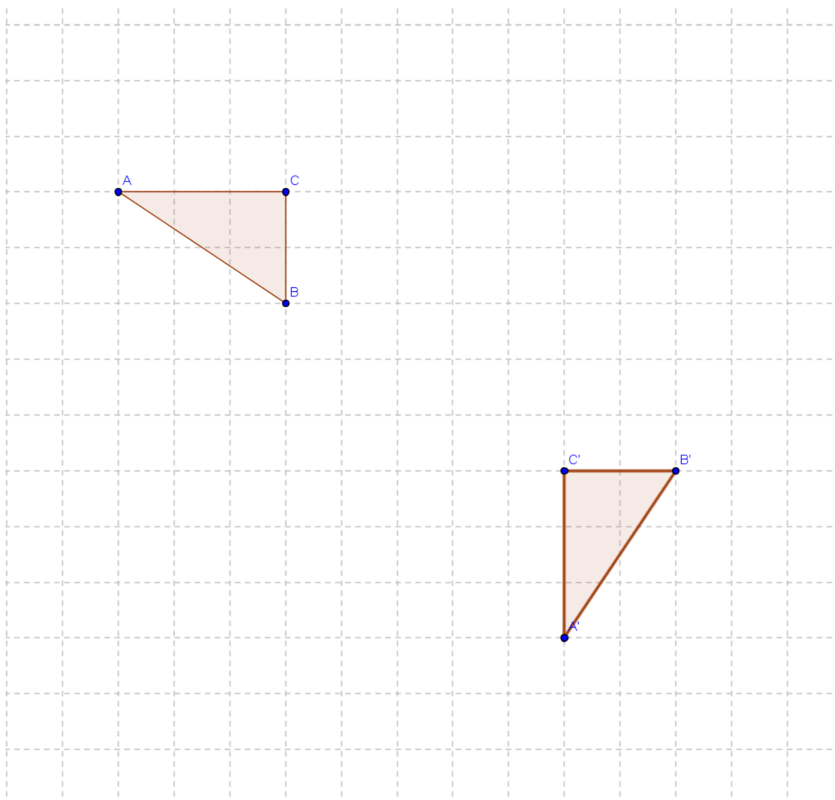
<u>Rotation</u> –

<u>Translation</u> –

State which isometry is used to transform these images and give either the line of reflection, vector of translation, or center and angle of rotation.




A
C
B
A'
C'
B'


A
C
B
C'
B'
A'

Lesson 2:

Rigid Motions as Functions

Objective: Students will generalize their knowledge of transformations and identify them as functions.

Assessment: Quiz on transformations as functions and their invariances.

Colorado State Standards: 4.1 a. ii, iii, vi, vii, b. i

Contents:

- **Warm Up 2.1**
- **Notes 2.1**
- **Quiz 2.1**
- **Warm Up 2.2**
- **In Class Exploration 2.2**
- **Quiz 2.2**

Warm Up 2.1

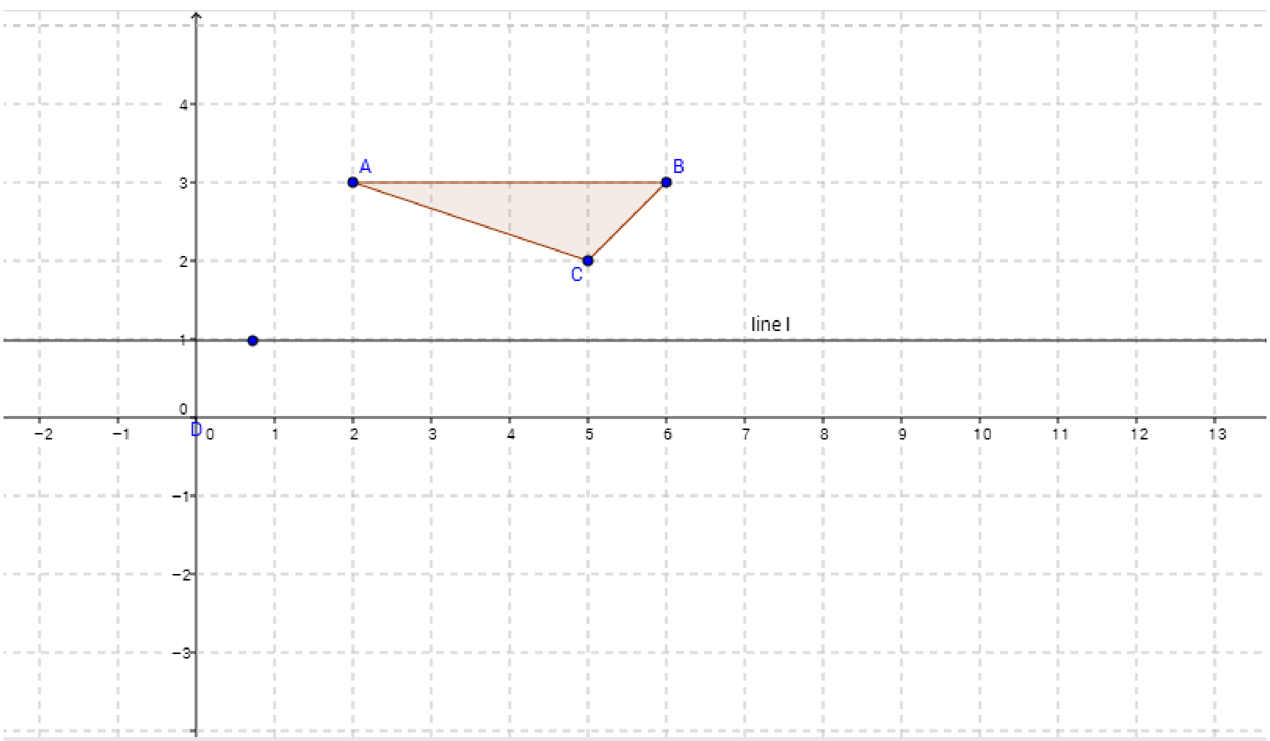
1) What is the definition of a function?

2) Define $f(x)=4x+7+x^2$. What is f(3)?

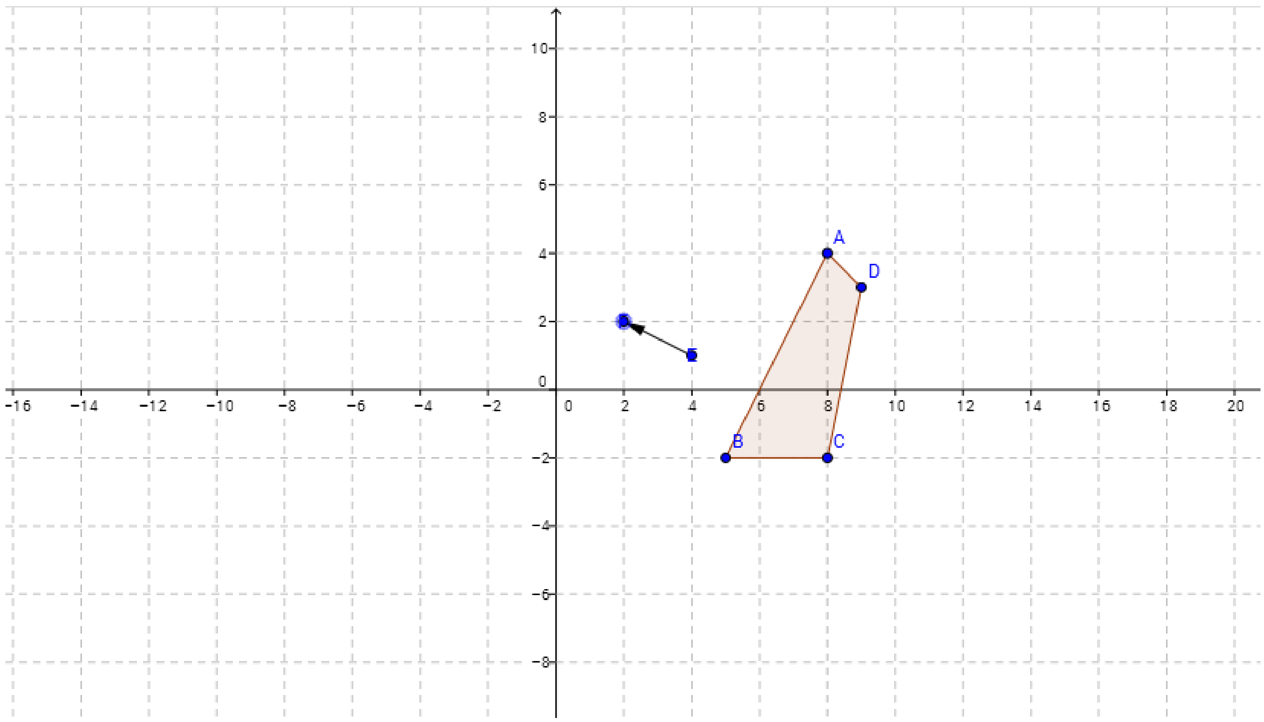
x	56	347	457	0
y	35	85	67	12

3) The table above defines a function r(x)=y. What is r(457)?

4) Reflect the given shape over line l.



5) Translate the given shape by the given vector.

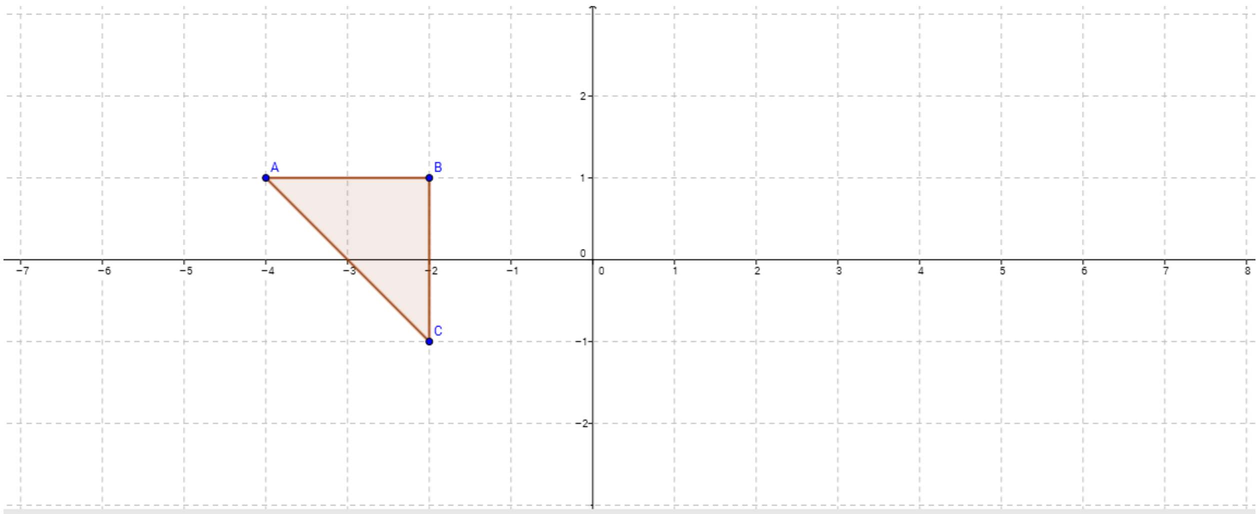


Notes 2.1

- Since this lesson is all about functions, the first thing that we will need to focus on is the function notation.
- Looking at the warm-up problem, think about how you solved it. You probably completed the same set of actions to each point on the given shape. (Talk about what a function is, draw the "function box", etc…. should all be review)
  - So, technically, I could give you any point and you could perform the given transformation on it, right? Give example of this.
  - Therefore, these transformations we have been talking about (reflection, rotation, translation) are all FUNCTIONS. They take a point as input, go into the "function box," and then spit out a new point as output.
- Think back to other functions we know about; $f(x)=3x+2$.
  - What happens if we want to know the output for an input value of 4? We write it like this…. We can also write these transformations in this way.
- Give a few examples of how it could look.
- Okay, so here is the thing. Mathematicians like to feel cool. So we use Greek letters instead of normal ones, specifically when we are talking about the type of functions called transformations.
  - Here are a few of the Greek letters we might be using.
    - Alpha α, Beta β, Gamma γ, Phi φ, Rho ρ, Sigma σ, Omega ω
  - On each letter, have the kids see/try a sample problem where they have to perform a transformation based on a problem you give them in function notation. Be sure to say the name of the letter and show how to write it many times.

Quiz 2.1

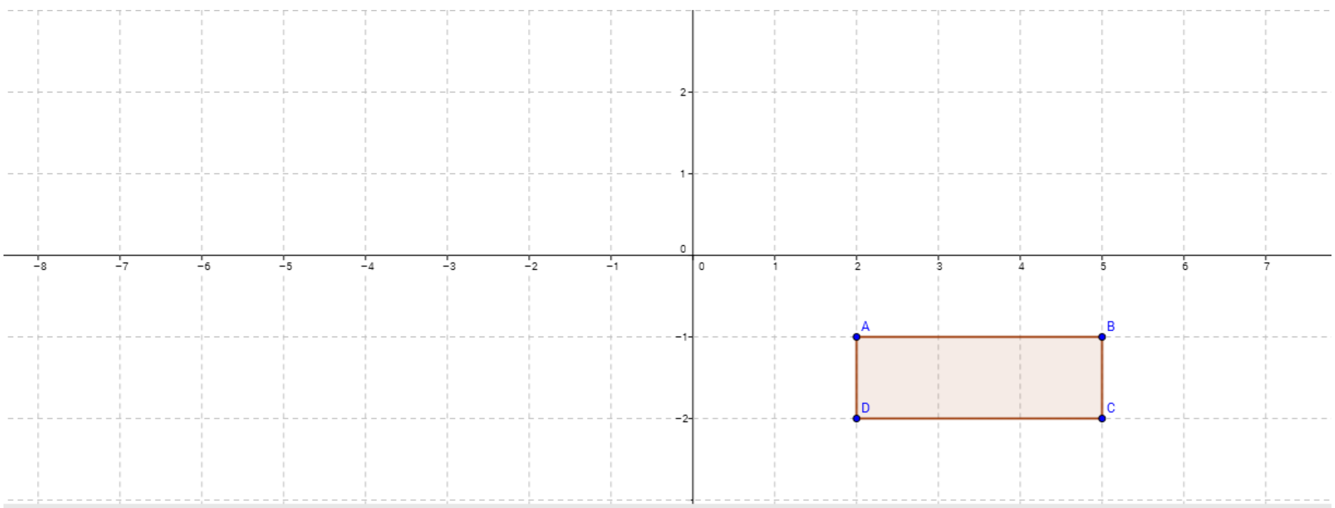
Transform the given triangle ABC by defining one reflection, one rotation and one translation using Greek letters and appropriate function notation. Draw the triangle after each transformation and label appropriately.



Warm Up 2.2

Turn to a table partner and tell them two of your favorite Greek letters. Try to remember (without looking at your notes!) how to write your partner's favorite letters. How many Greek letters can you and your partner come up with as a team?

Define ρ to be a rotation about the origin of 180°. What is ρ(ABCD)? Which Greek letter is defining this transformation?



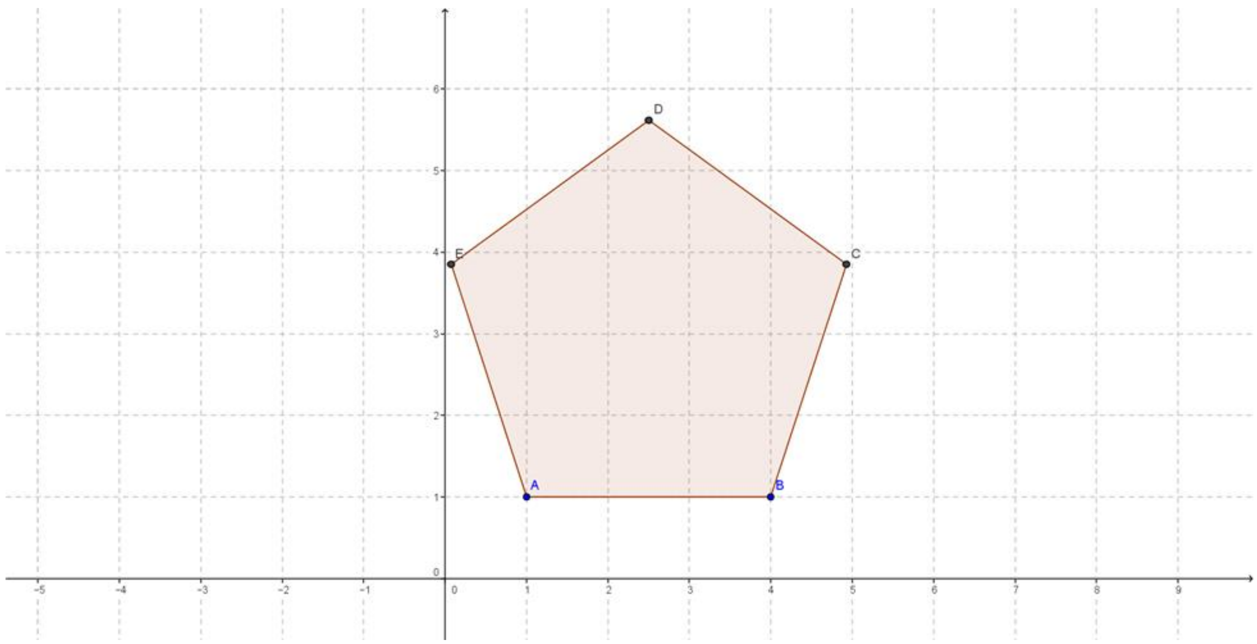
In-Class Exploration 2.2

Note: This activity can also be done using physical manipulatives and graph paper if GeoGebra software is not readily available in the classroom. GeoGebra is available as an app on most phones and tablets, so letting students use their own technology may also be an option.

- Divide class into 5 groups (or more depending on class size) ahead of time based on achievement on the previous day's quiz. Assign each group (based on achievement level) a shape: rectangle, parallelogram, trapezoid, regular pentagon, regular octagon (or other regular n-gons?)
- Students will create the given shape in GeoGebra
  - If needed, teacher can show how to use GeoGebra tools to create these shapes to students in small groups
- Students should, using trial and error, look for and take notes about transformations which carry the given shape onto itself
  - As the more advanced students start to see patterns, encourage them to generalize their findings to apply to other types of shapes.
  - Encourage students to find more creative reflections, rotations, and as many as possible
- After students have found how to map their shapes onto themselves, ask them to experiment with larger n-gons.
  - What is the difference between even and odd n-gons?
  - How can we generalize rotations which map a regular n-gon onto itself for any value of n?
  - Are there ever any vectors (other than the zero vector) which can map a shape to itself?
- As a class, generalize discoveries as notes into interactive notebook.

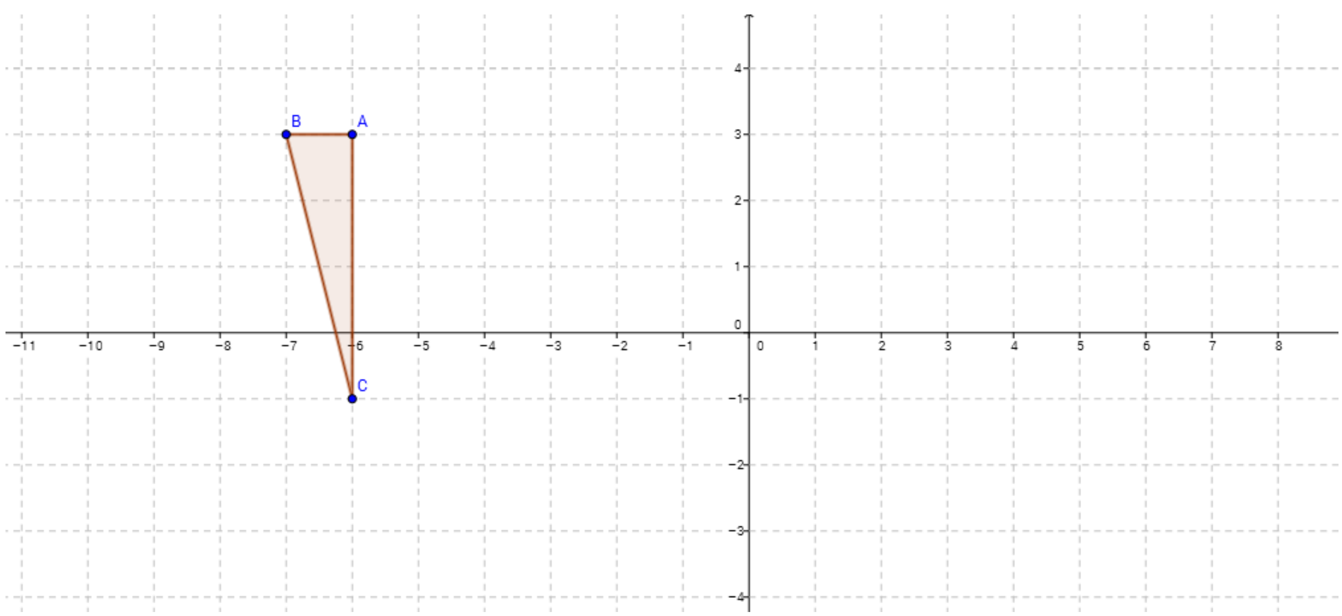
Quiz 2.2

Write, in function notation, at least 5 transformations that carry the given shape onto itself



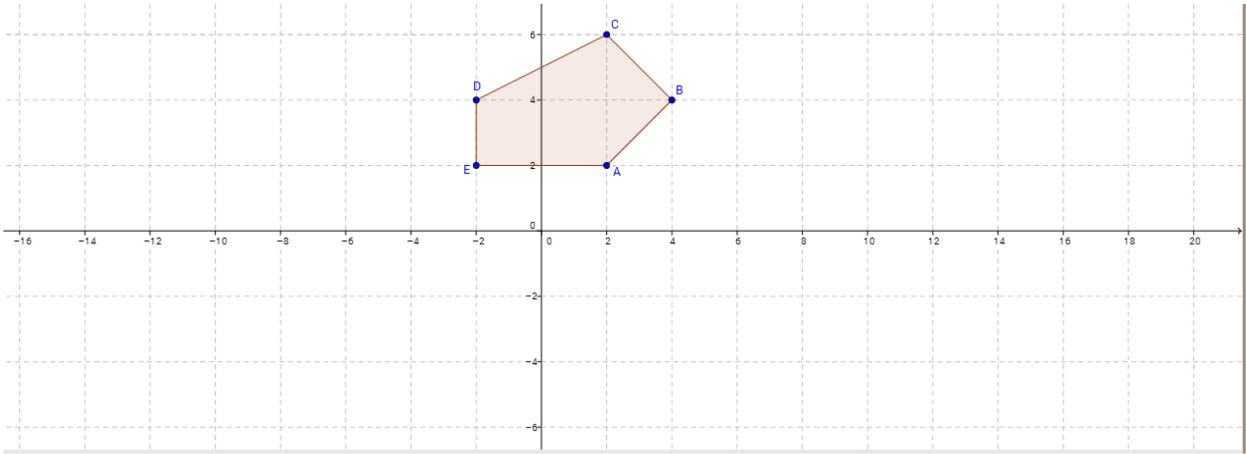
Apply the given function to the given shape.

Define β to be a reflection over the line y=0. What is β(ΔABC)?



Apply the given function to the given shape.

Define ω to be a translation by the vector [3,-6]. What is ω(ABCDE)?



Lesson 3:

Rigid Motion Composition

Objective: Students will compose isometries and identify them as a composition of reflections.

Assessment: Quiz on transformations as compositions of reflections.

Colorado State Standards: 4.1 a. iii, vii, viii

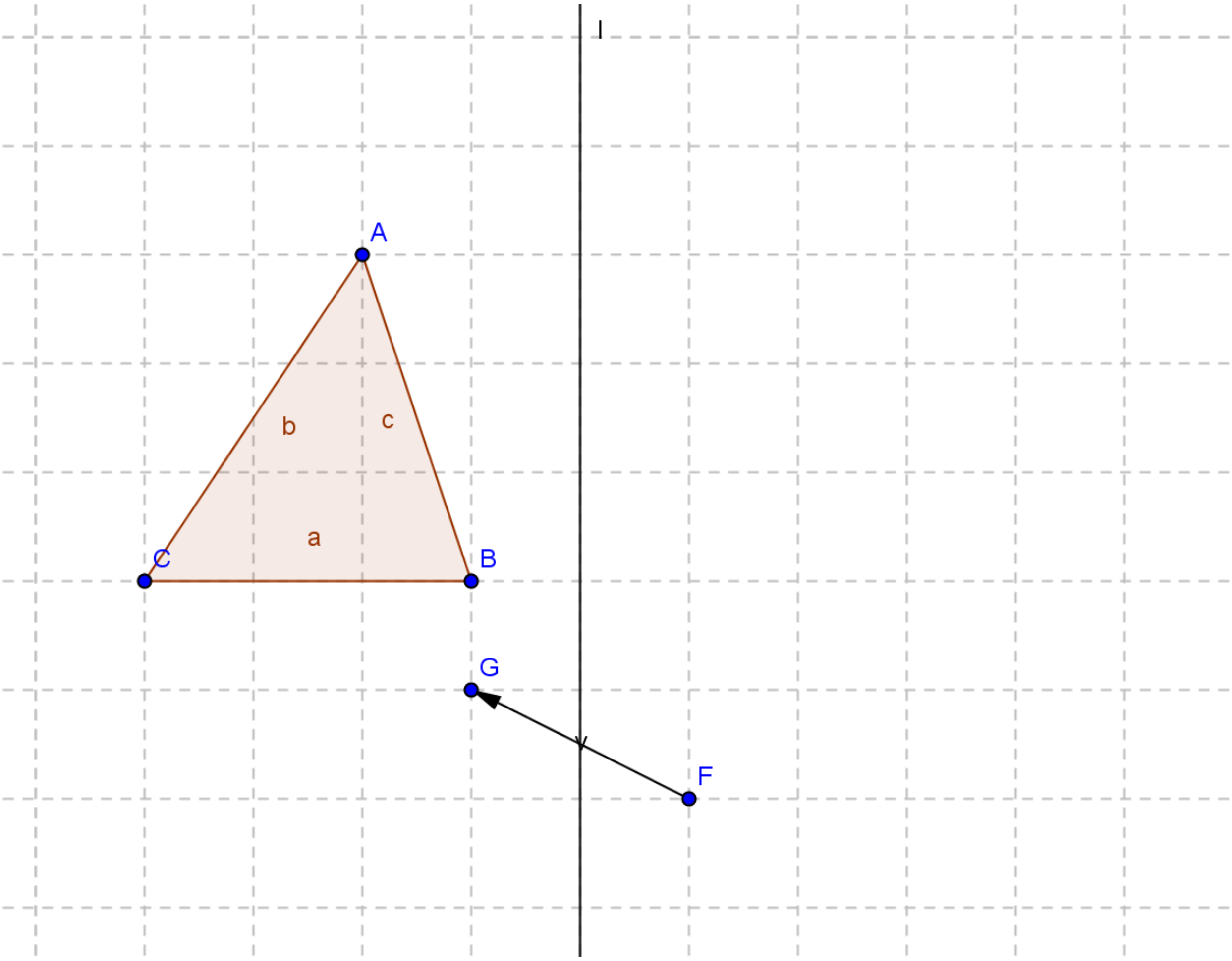
Contents:

- **Warm Up 3.1**
- **Notes 3.1**
- **In Class Exploration 3.1**
- **Warm Up 3.2**
- **Notes 3.2**
- **In Class Exploration 3.2**
- **Quiz 3.2**

Warm Up 3.1

Let σ be the reflection over the line l. Draw triangle A'B'C' the image of triangle ABC under σ.

Let φ be the translation by the vector v. Draw φ (A'B'C') and label it A"B"C".



Notes 3.1

General Function Composition

g(f(x)) is a composition of the functions g(x) and f(x). The notation indicates that x is an input of the function f(x), and that f(x) is the input of the function g(x).

g(f(x)) can also be written $g \circ f(x)$

In general, it is not true that g(f(x))= f(g(x)).

Examples

Ex. 1a

$g(x)=x^2$, $f(x)=3x-2$

What is g(f(2))?

$g(f(2))=g(3(2)-2)=g(4)=4^2=16$

Notice that we first evaluate the inner function.

Ex. 1b

What is f(g(2))?

$f(g(2))=f((2)^2)=f(4)=3(4)-2=10$

Note: $g(f(2))\neq f(g(2))$

Ex. 2

$g(x)=\sqrt{x}$, $f(x)=x^3+3$

What is g(f(1))?

$g(f(1))=g((1)^3+3)=g(4)=\sqrt{4}=2$

We learned that rigid motions are functions that map points and figures in the plane to other points and figures.

We can apply the concept of function composition to rigid motions. This means we first apply one rigid motion, and then the next.

Function notation from the warm up:

φ(σ(ΔABC))= φ (ΔA’B’C’)= ΔA”B”C”

This means that we first evaluate σ (ΔABC) to obtain ΔA’B’C’.

Then we evaluate φ (ΔA’B’C’) to obtain ΔA”B”C”.

This is another example of function composition.

Ex. 3

Let φ (ΔABC)=ΔA’B’C’.

Then let σ(ΔA’B’C’)= A”B”C”.

Draw σ(φ(ΔABC)).

Is this the same as φ (σ(ΔABC))?

In Class Exploration 3.1

Turn in all work done in GeoGebra as a printed Microsoft Word document.

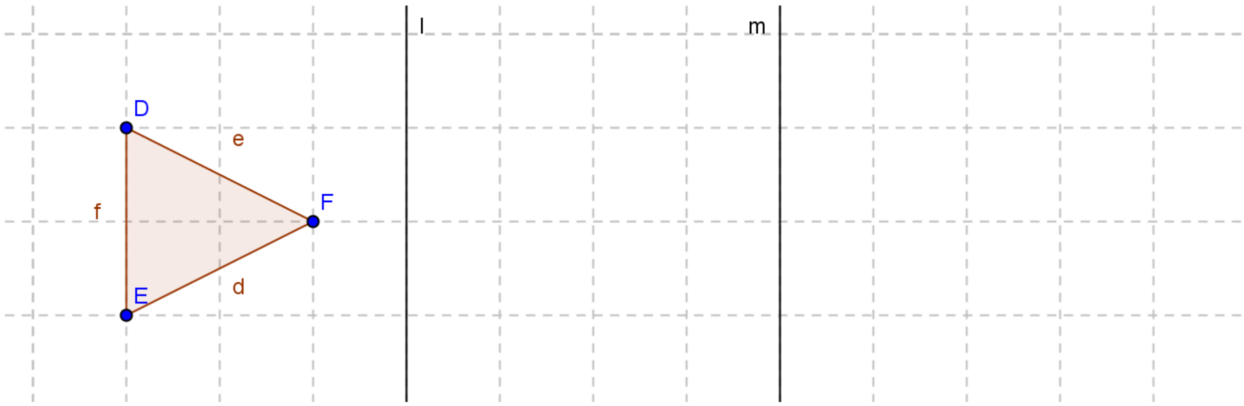
After completing the steps in GeoGebra for a problem, first click "Edit", then click on "Graphics View to Clipboard", and paste your work in your Microsoft Word document. Number your work.

For this assignment, you may ask classmates for help, but you are expected to turn in your own work.

1. **Class example:** In GeoGebra, draw ΔABC, a point D, and a line l. Let ρ be the clockwise 45° rotation about the point D. Let σ be the reflection over the line l.
   a. Find ΔA'B'C'=ρ(ΔABC).
   b. Find ΔA"B"C"=σ(ΔA'B'C').
2. In GeoGebra, draw Δ ABC, a vector v, and a point D. Let φ be the translation by v. Let ρ be the clockwise 68° rotation about the point D. Find Δ A"B"C"=φ(ρ(Δ ABC))
3. Draw Δ ABC, a point D, and a point E. Let ρ be the 45° clockwise rotation about D. Let ω be the 55° clockwise rotation about E. Find Δ A"B"C"= ω(ρ(Δ ABC)).
4. Draw a line l, a vector v, and a line m. Let φ be the translation by the vector v. Let σ be the reflection over m. Find l"= φ(σ(l)).
5. Draw the point A, the line l, and the line m, and the point B. Let σ be the reflection over l. Let μ be the reflection over m. Let ρ be the 30° clockwise rotation about A. Find A"'= μ(ρ(σ(A))).
6. Draw Δ ABC. Come up with two rigid motions α and β so that α(β(Δ ABC))= β(α(Δ ABC)). Describe α and β.

Warm up 3.2

Let α be the reflection over l and let β be the reflection over m. Draw A'B'C'= α (ABC). Draw A"B"C"= β (A'B'C'). Is the rigid motion β∘α the same as some other type of rigid motion?



It's the same as a translation. What is the vector it is being translated by?

Call this translation φ. Then φ(ABC)= β∘α(ABC)= β(α(ABC)). Actually these rigid motions are the same.

In GeoGebra, draw triangle ABC, a line l, and lines m and n both parallel to l. Let α be the reflection over l, β be the reflection over m, and γ the reflection over n. Draw A"'B"'C"'= γ(β(α(ABC))). What kind of rigid motion is γ∘β∘α?

Reflection, notice the orientation. How do we find the line of reflection?

Notes 3.2

The composition from the warm-up is the same as a translation. By what vector is it being translated?

Call this translation φ. Then φ(Δ ABC)= β∘α(Δ ABC)= β(α (Δ ABC)). These rigid motions are the same.

In GeoGebra, draw Δ ABC, a line l, and lines m and n both parallel to l. Let α be the reflection over l, β be the reflection over m, and γ the reflection over n. Draw Δ A'''B'''C'''= γ(β(α(Δ ABC))). What kind of rigid motion is γ∘β∘α?

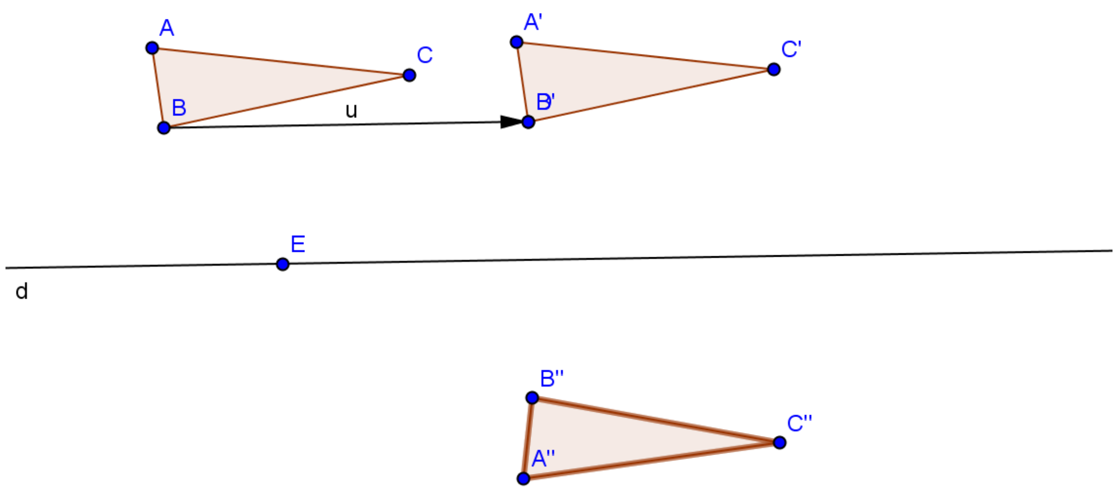
γ∘β∘α is a reflection. Notice that the orientations of the ΔABC and ΔA'''B'''C''' are opposite. How do we find the line of reflection?

Connect the triangles vertices. The line of reflection should be the perpendicular bisector of all of the lines.

Glide Reflections

A glide reflection is the composition of a translation and a reflection across a line parallel to the translation vector (626).

This is also a rigid motion since it's a composition of rigid motions.



In Class Exploration 3.2

Turn in all work done in GeoGebra as a printed Microsoft Word document.

After completing the steps in GeoGebra for a problem, first click “Edit”, then click on “Graphics View to Clipboard”, and paste your work in your Microsoft Word document. Number your work.

For this assignment, you may ask classmates for help, but you are expected to turn in your own work

For each problem,

Apply the rigid motion that is a composition of reflections in lines to ΔABC to find the image ΔABC. (If the rigid motion is a composition of two reflections, the image will be labeled ΔA’B’C’. If it is a composition of three reflections, the image will be labeled ΔA’’B’’C’’.)

a) State whether the orientation of the image of ΔABC under the rigid motion is the same or different from that of ΔABC.
b) Determine if the composition of reflections is a translation, a rotation, a reflection, or a glide reflection. State how you can tell.

1. Two parallel lines (as a class)
   a)

   b)

2. Three parallel lines (as a class)
   a)

   b)

3. Two lines that meet in one point
   a)

   b)

4. Three lines that meet in one point

   a)

   b)

5. Three lines that form a triangle

   a)

   b)

6. Two parallel lines and another line not parallel to the first two

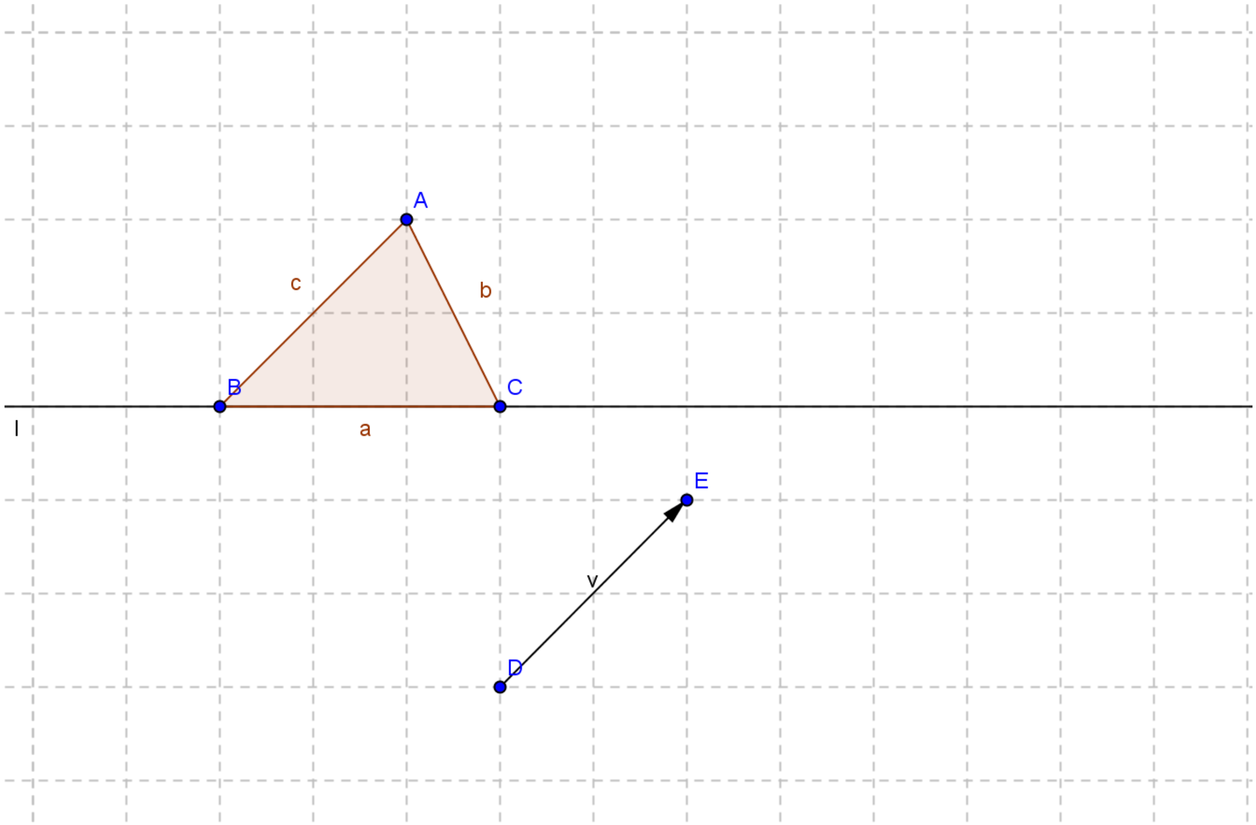
   a)

   b)

* For previous activity, review answers with **1_composition_of_reflections.ggb** and **2_composition_of_reflections.ggb.**

Quiz 3.2

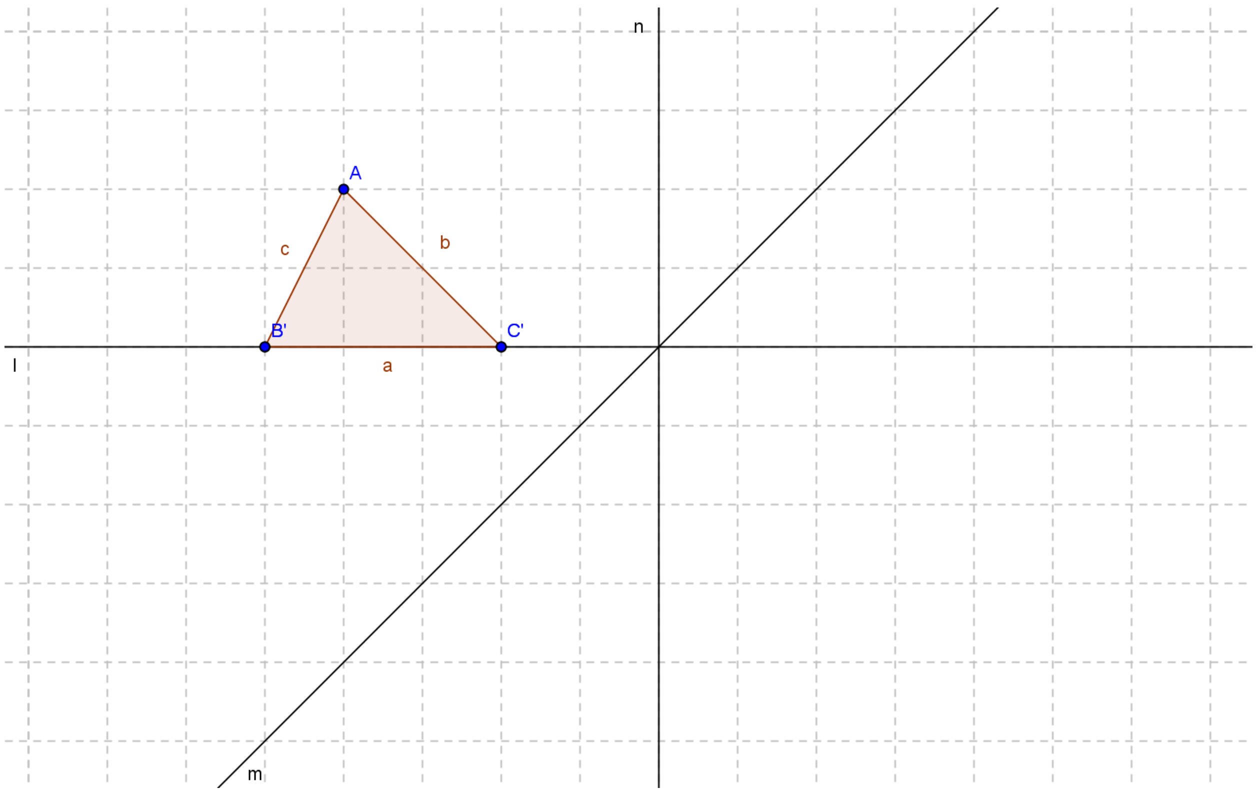
1. Let σ be the reflection over l. Let φ be the translation by v. Draw Δ A"B"C"=φ(σ(Δ ABC)). Is φ∘σ= σ ∘ φ? How do you know?



2. Write down the type of rigid motion that is equivalent to the composition of reflections over the following types of lines:

1) 2 parallel lines
2) 1 line
3) 3 parallel lines
4) 2 lines that intersect at one point
5) 3 lines that intersect at one point
6) 3 lines that form a triangle

3. Consider the three lines l,m, and n that meet at a point. Let α be the reflection over l, let β be the reflection over m, and let γ be the reflection over n.
Draw ΔA'''B'''C'''= γ(β (α(ΔABC))). What kind of rigid motion is γ∘β∘α?



Lesson 4:

The Three Reflections Theorem

Objective: Students will apply transformations to proving the three reflection theorem using GeoGebra.

Assessment: Written response quiz on the three-reflections theorem.

Colorado State Standards: 4.1 a. viii, b. i, ii, c. i.

Contents:

- **Warm Up 4.1**
- **Notes 4.1**
- **In Class Exploration 4.1**
- **Warm Up 4.2**
- **In Class Exploration 4.2**
- **Quiz 4.2**
- **Answer Key 4**

Warm up 4.1

Write down the type of rigid motion that is equivalent to the composition of reflections over the following types of lines:

1) 2 parallel lines
2) 1 line
3) 3 parallel lines
4) 2 concurrent lines
5) 3 concurrent lines
6) 3 lines that form a triangle

Notes 4.1

We can also think about the composition of reflections over zero lines. We call this function the identity. The identity maps every point to itself, so it does not move anything in the plane. It is the “do nothing” transformation. The identity is a rigid motion because it clearly preserves length and angle measure.

We know that the identity, reflections, translations, rotations, and glide reflections can be expressed as a composition of reflections. For each of the cases we covered, we used compositions of at most three lines.

Some questions to consider:

1. Are the four rigid motions we know about the only kinds?

2. Are there any rigid motions that are the same as a composition of reflections over more than three lines?

Yes, these are the only possible rigid motions.

We can create a rigid motion by composing reflections over as many lines as we want, but it turns out that the resulting rigid motion can be expressed as a composition of at most three lines.

Show example in GeoGebra. Use **3_translation_example.ggb**

In Class Exploration 4.1

Preliminary Statement-Reason Proofs

*A key of the statement-reason proofs is provided at the end of this lesson.

Students will carry out constructions and apply rigid motions in GeoGebra. They will fill out missing portions of each Statement-Reason proof.

Class example

Show in GeoGebra. Use **4_perpendicular_equidistant.ggb**

Proof 1

Points on a Line Segment's Perpendicular Bisector are Equidistant from the Line Segment's Endpoints

Statement	Reason
1. Let $\overline{AB}$ be an arbitrary line segment.	Construction
2. Let b be $\overline{AB}$'s perpendicular bisector.	Construction
3. Let C be an arbitrary point on b.	Construction
4. Draw $\overline{AC}$.	Construction
5. The reflection over b maps C to ___	
6. The reflection over b maps A to ___	
7. The reflection over b maps $\overline{AC}$ to ___	The reflection over b maps A to __ and C to __ .
8. $\overline{AC} \cong$ ____	The reflection over b maps $\overline{AC}$ to ___, and rigid motions preserve length.
9. C is equidistant from ___ and ___.	$\overline{AC} \cong$ ___
10.	$\overline{AB}$ is an arbitrary line segment (1), and C is an arbitrary point on $\overline{AB}$'s perpendicular bisector (3).

Ray theorem: If the angle between two rays with the same vertex is bisected by a line, then the reflection over the angle bisector interchanges the rays.

Work in pairs or groups.

Proof 2

Statement	Reason
1. Consider two arbitrary rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.	Construction
2. Let l be the angle bisector of ∠BAC.	Construction
3. Let D be a point on the interior of ∠BAC on l.	Construction
4. ∠BAD≅ ∠DAC	
5. A is on l.	
6. The reflection over l maps A to ___.	
7. The reflection over l maps D to ____.	
8. The reflection over l maps $\overline{AD}$ to _____.	Reflection over l maps A to___ and D to ___.
9. Reflection over l maps ∠BAD to ______..	Reflection over l maps B to ___, A to ___, and D to ___.
10. ∠BAD≅ ______	Reflection over l maps ∠BAD maps to ____, and rigid motions preserve angle measure.
11. ∠B'AD≅______	∠BAD≅______, ∠BAD≅______, and congruence of angle measure is transitive.
12. B' is on ____.	B' is on the opposite side of l from B since B' is B reflected over l, and B' must be a point that satisfies ∠B'AD≅________.
13. Reflection over l maps $\overrightarrow{AB}$ to ____.	Reflection over l maps A to ___, and B maps to ___, a point on $\overrightarrow{AC}$ (12).
14. The image of $\overrightarrow{AC}$ is _____.	Similarly, the image of $\overrightarrow{AB}$ is $\overrightarrow{AC}$.
15.	Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with the same vertex are arbitrary vectors. Reflection over l, the angle bisector, interchanges $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Review with students, use **5_ray_theorem.ggb**

Work in pairs or groups.

Proof 3

Points Equidistant from a Line Segment's Endpoints are on the Perpendicular Bisector of the Segment

Statement	Reason
1. Let $\overline{AB}$ be an arbitrary line segment.	Construction
2. Let C be an arbitrary point equidistant from A and B.	Construction
3. $\overline{CA} \cong$ ____	
4. Let d be the angle bisector of ∠ACB.	Construction
5. C is on d.	
6. d bisects the angle formed by $\overrightarrow{CA}$ and $\overrightarrow{CB}$.	d bisects ______, and ∠ACB is the angle formed by $\overrightarrow{CA}$ and _____.
7. Reflection over d maps $\overrightarrow{CA}$ to_____.	
8. Reflection over d maps A to ___.	The image of A is on _____ and the image of A must be a distance $\lvert\overline{CA}\rvert = \lvert\overline{CB}\rvert$ from C (3), which corresponds to the point ___.
9. d is $\overline{AB}$'s perpendicular bisector.	
10. C is on $\overline{AB}$'s perpendicular bisector.	C is on ___, and d is $\overline{AB}$'s _______________.
11.	$\overline{AB}$ is an arbitrary line segment (1), and the point C is an arbitrary point equidistant from A and B (2).

Review results with students and show in GeoGebra. Use **6_equidistant_perpendicular.ggb**

Warm Up 4.2

Write what the reflection-theorem says in your own words. You may use your notes from yesterday.

In GeoGebra, show how a composition of reflections in four parallel lines can be reduced to a composition of reflection in two parallel lines.

In Class Exploration 4.2

Explain how to randomly construct triangles with **7_congruent_triangles.ggb**

Use **8_threereflections_sameorientation.ggb** and **9_threereflections_oppositeorientation.ggb**.

Three-Reflections Theorem Activity

Save all work in GeoGebra in a Microsoft Word document. After completing the steps in GeoGebra for a problem, click "Edit", click on "Graphics View to Clipboard", go to your Microsoft Word document, and right click and paste.

Case 1: Triangles of the Same Orientation

Carry out constructions and apply rigid motions in GeoGebra. Fill out missing portions of the Statement-Reason proof.

Statement	Reason
1. Let ΔABC be an arbitrary triangle.	Construction
2. Randomly choose ΔA'B'C' congruent to ΔABC with the same orientation.	Construction
3. $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, $\overline{BC} \cong \overline{B'C'}$ ∠ABC ≅ ∠A'B'C', ∠BAC ≅ ∠B'A'C', ∠ACB ≅ ∠A'C'B'	
4. Draw $\overline{AA'}$.	Construction
5. Draw e, the perpendicular bisector of $\overline{AA'}$.	Construction
6. Reflect ΔA'B'C' over e.	Transformation
7. Reflection over e maps A' to ___.	
8. Reflection over e maps $\overline{A'B'}$ to _____.	Reflection over e maps A'to___ and B' to ___.
9. Reflection over e maps $\overline{A'C'}$ to _____.	Reflection over e maps A' to ___ and C' to ___.
10. Reflection over e maps ∠B'A'C' to ________.	Reflection over e maps B to ___, A' to ___, and C' to C''.
11. Draw $\overline{BB''}$.	Construction
12. Draw g, the perpendicular bisector of $\overline{BB''}$.	Construction
13. Reflect ΔAB''C'' over g.	Transformation
14. Reflection over g maps B'' to ___.	

15. $\overline{A'B'} \cong$_____	Reflection over e maps $\overline{A'B'}$ to ______, and rigid motions preserve length.
16. $\overline{AB} \cong$_____	_________, __________, and congruence of line segments is transitive.
17. A is on g.	$\overline{AB} \cong$ ______, so A is equidistant from B and B'' by definition of congruence, thus A is on $\overline{BB''}$'s _______________.
18. Reflection over g maps A to A.	
19. ∠B''AC''≅________	Reflection over e maps ∠B'A'C' to _______, and rigid motions preserve angle measure.
20. ∠B''AC''≅_________	∠B''AC''≅ ______, ∠B'A'C'≅ ________, and congruence of angles is transitive.
21. Let D be a point on g in the interior of ∠CAC.	Construction
22. Reflection over g maps D to D.	
23. Reflection over g maps ∠B''AD to ∠BAD.	Reflection over g maps B'' maps to B (14), A maps to A (18), and D maps to D (22).
24. ∠B''AD ≅ _______	Reflection over g maps ∠B''AD to ______.
25. ∠C''AD≅ ________	∠B''AD ≅ ______, ∠B''AC''≅ ______, and subtraction/ addition of angles.
26. g bisects the angle formed by the rays $\overrightarrow{AC''}$ and _____.	∠C''AD ≅ _______, A and D are on g (17,21), and $\overrightarrow{AC''}$ and $\overrightarrow{AC}$ are on opposite sides of g since the orientations of ΔABC and ΔAB''C'' are ___________ after one reflection.
27. Reflection over g maps $\overrightarrow{AC''}$ to _____.	
28. $\overline{AC''} \cong \overline{A'C'}$	Reflection over e maps $\overline{A'C'}$ to ______, and rigid motions preserve length.
29. $\overline{AC''} \cong$ _____	$\overline{AC''} \cong$______, $\overline{A'C'} \cong$______, and congruence of line segments is transitive.

30. Reflection over g maps C'' to ___.	The image of C'' under reflection over g must be on _____, and it must be a distance $\left\|\overline{AC''}\right\|$ = $\left\|\overline{\_\_\_}\right\|$ from A , which corresponds to the point ___.
31.	ΔABC is an arbitrary triangle, and ΔA'B'C' is randomly chosen such that it is congruent to ΔABC with the same orientation. We obtain ΔABC from ΔA'B'C' by reflecting ΔA'B'C' over two concurrent lines. (This is a rotation).

Case 2: Triangles of the Opposite Orientation

Work in groups or pairs. Carry out constructions and apply rigid motions in GeoGebra. Fill out missing portions of the Statement-Reason proof.

Statement	Reason
1. Let ΔABC be an arbitrary triangle.	Construction
2. Randomly choose ΔA'B'C' congruent to ΔABC with the opposite orientation.	Construction
3. $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, $\overline{BC} \cong \overline{B'C'}$ ∠ABC ≅ ∠A'B'C', ∠BAC ≅ ∠B'A'C', ∠ACB ≅ ∠A'C'B'	
4. Draw $\overline{AA'}$.	Construction
5. Draw f, the perpendicular bisector of $\overline{AA'}$.	Construction
6. Reflection over f maps A' to___.	
7. Reflection over f maps $\overline{A'B'}$ to____.	Reflection over f maps A' to ___ and B' to ___.
8. Reflection over f maps ∠A'B'C' to _____.	Reflection over f maps A' to ___, B' to ___, and C' to ____.
9. Reflection over f maps $\overline{B'C'}$ to _____.	Reflection over f maps B' to ___ and C' to ___.
10. Draw $\overline{BB''}$.	Construction
11. Draw h, the perpendicular bisector of $\overline{BB''}$.	Construction
12. Reflection over h maps B'' to ___.	
13. $\overline{A'B'} \cong \overline{AB''}$	The reflection over f maps $\overline{A'B'}$ to____ , and rigid motions preserve length.
14. $\overline{AB} \cong \overline{AB''}$	________, ________, and congruence of line segments is transitive.
15. A is on h.	________, so A is equidistant from B and B'' by definition of congruence. Thus, A is on $\overline{BB''}$'s _______________________.
16. Reflection over h maps A to ____.	
17. Reflection over h maps ∠AB''C'' to ________.	Reflection over h maps A to ___, B'' to ____, and C'' to C'''.
18. Reflection over h maps $\overline{AB''}$ to _____.	Reflection over h maps A to ___ and maps B'' to ____.
19. Reflection over h maps $\overline{B''C''}$ to _____.	Reflection over h maps B'' to ____ and C'' to C'''.

20. Let j be the line containing A and B.	Construction
21. $\overline{AB}$ is on j.	
22. Reflecting over j maps $\overline{AB}$ to ____.	
23. The composition of reflections over f and h maps ∠A'B'C' to _______.	Reflection over f maps ∠A'B'C' to ________, and reflection over h maps ∠AB''C'' to ________.
24. ∠A'B'C' ≅ ∠ABC'''	The composition of reflections over f and h maps ∠A'B'C' to ________, and rigid motions preserve angle measure.
25. ∠ABC ≅ ∠ABC'''	________,________, and congruence of angles is transitive.
26. j is the angle bisector of the angle formed by the rays $\overrightarrow{BC'''}$ to $\overrightarrow{BC}$.	∠ABC ≅ ______, $\overline{AB}$ is on j (21), and $\overrightarrow{BC'''}$ and $\overrightarrow{BC}$ are on opposite sides of j since the orientation of the triangles is ________ after two reflections.
27. Reflection over j maps $\overrightarrow{BC'''}$ to ____.	j is the angle bisector of the angle formed by the rays $\overrightarrow{BC'''}$ to $\overrightarrow{BC}$.
28. The composition of reflections over f and h maps $\overline{B'C'}$ to _____.	Reflection over f maps $\overline{B'C'}$ to _____, and reflection over h maps $\overline{B''C''}$ to _____.
29. $\overline{B'C'} \cong \overline{BC'''}$	The composition of reflections over f and h maps $\overline{B'C'}$ to_____, and rigid motions preserve length.
30. $\overline{BC} \cong \overline{BC'''}$	__________,__________, and congruence of line segments is transitive.
31. C''' maps to ____.	The image of C''' under reflection over j must be on and it must be a distance $\left\lvert\overline{BC'''}\right\rvert =$ $\left\lvert\underline{\quad}\right\rvert$ from B (30), which corresponds to the point ___.
32.	ΔABC is an arbitrary triangle and ΔA'B'C' is randomly chosen such that it is congruent to ΔABC with the opposite orientation. We obtain ΔABC from ΔA'B'C' by reflecting ΔA'B'C' over three lines that are not all parallel or all concurrent. (This is a glide reflection).

Quiz 4.2

Answer the following questions in sentences. You may refer to your Statement-Reason proofs.

1) If you must use a reflection to a point A to another point A', what line must you reflect over? Why must reflecting over this line map A to A'?
2) In proving the three reflections theorem, we often used the fact that a point A was equidistant from the endpoints of a line segment to prove what about the location of A?
3) Why didn't C'' map to C in Case 2 like it did in Case 1?
4) Given two randomly selected congruent triangles with the same orientation, we mapped one triangle to another in how many reflections? What kind of rigid motion was this composition of reflections?
5) Given two randomly selected congruent triangles with the opposite orientation, we mapped one triangle to another in how many reflections? What kind of rigid motion was this composition of reflections?

Answer Key 4

Points on a Line Segment's Perpendicular Bisector are Equidistant from the Line Segment's Endpoints

Statement	Reason
1. Let $\overline{AB}$ be an arbitrary line segment.	Construction
2. Let b be $\overline{AB}$'s perpendicular bisector.	Construction
3. Let C be an arbitrary point on b.	Construction
4. Draw $\overline{AC}$.	Construction
5. The reflection over b maps C to C.	C is on b (3).
6. The reflection over b maps A to B.	A and B are equidistant from b on a line segment perpendicular to b (2).
7. The reflection over b maps $\overline{AC}$ to $\overline{BC}$	The reflection over b maps A to B (6) and C to C (3).
8. $\overline{AC} \cong \overline{BC}$	The reflection over b maps $\overline{AC}$ to $\overline{BC}$ (7), and rigid motions preserve length.
9. C is equidistant from A and B.	$\overline{AC} \cong \overline{BC}$ (7)
10. Any point on a line segment's perpendicular bisector is equidistant from the line segment's end points.	$\overline{AB}$ is an arbitrary line segment (1), and C is an arbitrary point on $\overline{AB}$'s perpendicular bisector (3).

Ray Theorem: If the angle between two rays with the same vertex is bisected by a line, then the reflection over the angle bisector interchanges the rays.

Statement	Reason
16. Consider two arbitrary rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.	Construction
17. Let l be the angle bisector of ∠BAC.	Construction
18. Let D be a point on the interior of ∠BAC on l.	Construction
19. ∠BAD≅ ∠DAC	D is on l (3), and l is ∠BAC's angle bisector (2).
20. A is on l.	l is the angle bisector of ∠BAC (2).
21. The reflection over l maps A to A.	A is on l (5).
22. The reflection over l maps D to D.	D is on l (3).
23. The reflection over l maps $\overline{AD}$ to $\overline{AD}$.	Reflection over l maps A to A (6) and D to D (7).
24. Reflection over l maps ∠BAD to ∠B'AD.	Reflection over l maps B to B', A to A (6), and D to D (7).
25. ∠BAD≅ ∠B'AD	Reflection over l maps ∠BAD maps to ∠B'AD, and rigid motions preserve angle measure.
26. ∠B'AD≅ ∠DAC	∠BAD≅ ∠B'AD (10), ∠BAD≅ ∠DAC (4), and congruence of angle measure is transitive.
27. B' is on $\overrightarrow{AC}$.	B' is on the opposite side of l from B since B' is B reflected over l, and B' must be a point that satisfies ∠B'AD≅ ∠DAC (11).
28. Reflection over l maps $\overrightarrow{AB}$ to $\overrightarrow{AC}$.	Reflection over l maps A to A (6), and B maps to B', a point on $\overrightarrow{AC}$ (12).
29. The image of $\overrightarrow{AC}$ is $\overrightarrow{AB}$.	Similarly, the image of $\overrightarrow{AB}$ is $\overrightarrow{AC}$.
30. If the angle between two rays with the same vertex is bisected by a line, then the reflection over the angle bisector interchanges the rays.	Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with the same vertex are arbitrary vectors. Reflection over l, the angle bisector, interchanges $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Points Equidistant from a Line Segment's Endpoints are on the Perpendicular Bisector of the Segment

Statement	Reason
12. Let $\overline{AB}$ be an arbitrary line segment.	Construction
13. Let C be an arbitrary point equidistant from A and B.	Construction
14. $\overline{CA} \cong \overline{CB}$	C is equidistant from A and B (2).
15. Let d be the angle bisector of ∠ACB.	Construction
16. C is on d.	d bisects ∠ACB (4).
17. d bisects the angle formed by $\overrightarrow{CA}$ and $\overrightarrow{CB}$.	d bisects ∠ACB (4), and ∠ACB is the angle formed by $\overrightarrow{CA}$ and $\overrightarrow{CB}$.
18. Reflection over d maps $\overrightarrow{CA}$ to $\overrightarrow{CB}$.	d bisects the angle formed by $\overrightarrow{CA}$ and $\overrightarrow{CB}$ (6).
19. Reflection over d maps A to B.	The image of A is on $\overrightarrow{CB}$ (7) and the image of A must be a distance $\lvert\overline{CA}\rvert = \lvert\overline{CB}\rvert$ from C (3), which corresponds to the point B.
20. d is $\overline{AB}$'s perpendicular bisector.	Reflection over d maps A to B (8).
21. C is on $\overline{AB}$'s perpendicular bisector.	C is on d (5), and d is $\overline{AB}$'s perpendicular bisector (9).
22. Points equidistant from a line segment's end points are on that line segment's perpendicular bisector.	$\overline{AB}$ is an arbitrary line segment (1), and the point C is an arbitrary point equidistant from A and B (2).

Three-Reflections Theorem Case 1

Statement	Reason
32. Let ΔABC be an arbitrary triangle.	Construction
33. Randomly choose ΔA’B’C’ congruent to ΔABC with the same orientation.	Construction
34. $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, $\overline{BC} \cong \overline{B'C'}$ ∠ABC ≅ ∠A’B’C’, ∠BAC ≅ ∠B’A’C’, ∠ACB ≅ ∠A’C’B’	ΔA’B’C’≅ ΔABC (2), and corresponding parts of congruent triangles are congruent.
35. Draw $\overline{AA'}$.	Construction
36. Draw e, the perpendicular bisector of $\overline{AA'}$.	Construction
37. Reflect ΔA’B’C’ over e.	Transformation
38. Reflection over e maps A’ to A.	e is the perpendicular bisector of $\overline{AA'}$ (5), and apply the definition of reflection.
39. Reflection over e maps $\overline{A'B'}$ to $\overline{AB''}$.	Reflection over e maps A’ to A (7) and B’ to B’’.
40. Reflection over e maps $\overline{A'C'}$ to $\overline{AC''}$.	Reflection over e maps A’ to A (7) and C’ to C’’.
41. Reflection over e maps ∠B’A’C’ to ∠B’’AC’’.	Reflection over e maps B to B’’, A’ to A (7), and C’ to C’’.
42. Draw $\overline{BB''}$.	Construction
43. Draw g, the perpendicular bisector of $\overline{BB''}$.	Construction
44. Reflect ΔAB’’C’’ over g.	Transformation
45. Reflection over g maps B’’ to B.	g is the perpendicular bisector of $\overline{BB''}$(12), and apply the definition of reflection.
46. $\overline{A'B'} \cong \overline{AB''}$	Reflection over e maps $\overline{A'B'}$ to $\overline{AB''}$(8), and rigid motions preserve length.
47. $\overline{AB} \cong \overline{AB''}$	$\overline{A'B'} \cong \overline{AB''}$ (15), $\overline{AB} \cong \overline{AB''}$ (3), and congruence of line segments is transitive.
48. A is on g.	$\overline{AB} \cong \overline{AB''}$ (13), so A is equidistant from B and B’’ by definition of congruence, thus A is on $\overline{BB''}$’s perpendicular bisector.
49. Reflection over g maps A to A.	A is on g (17).
50. ∠B’’AC’’≅ ∠B’A’C’	Reflection over e maps ∠B’A’C’ to ∠B’’AC’’ (10), and rigid motions preserve angle measure.

51. ∠B’’AC’’≅ ∠BAC	∠B’’AC’’≅ ∠B’A’C’ (19), ∠B’A’C’≅ ∠BAC (3), and congruence of angles is transitive.
52. Let D be a point on g in the interior of ∠CAC.	Construction
53. Reflection over g maps D to D.	D is on g (21).
54. Reflection over g maps ∠B’’AD to ∠BAD.	Reflection over g maps B’’ maps to B (14), A maps to A (18), and D maps to D (22).
55. ∠B’’AD ≅ ∠BAD	Reflection over g maps ∠B’’AD to∠BAD (23).
56. ∠C’’AD≅ ∠CAD	∠B’’AD ≅ ∠BAD (24), ∠B’’AC’’≅ ∠BAC (20), and subtraction/ addition of angles.
57. g bisects the angle formed by the rays $\overrightarrow{AC''}$ and $\overrightarrow{AC}$.	∠C’’AD ≅ ∠CAD, A and D are on g (17,21), and $\overrightarrow{AC''}$ and $\overrightarrow{AC}$ are on opposite sides of g since the orientations of ΔABC and ΔAB’’C’’ are opposite after one reflection.
58. Reflection over g maps $\overrightarrow{AC''}$ to $\overrightarrow{AC}$.	g bisects the angle formed by the rays $\overrightarrow{AC''}$ and (26).
59. $\overline{AC''} \cong \overline{A'C'}$	Reflection over e maps $\overline{A'C'}$ to $\overline{AC''}$ (9), and rigid motions preserve length.
60. $\overline{AC''} \cong \overline{AC}$	$\overline{AC''} \cong \overline{A'C'}$ (28), $\overline{A'C'} \cong \overline{AC}$ (3) , and congruence of line segments is transitive.
61. Reflection over g maps C’’ to C.	The image of C’’ under reflection over g must be on $\overrightarrow{AC}$ (26), and it must be a distance $\left\lvert \overline{AC''} \right\rvert =$ $\left\lvert \overline{AC} \right\rvert$ (29) from A , which corresponds to the point C.
62. For any two congruent triangles with the same orientation, one triangle can be mapped to the other under a composition of two reflections.	ΔABC is an arbitrary triangle, and ΔA’B’C’ is randomly chosen such that it is congruent to ΔABC with the same orientation. We obtain ΔABC from ΔA’B’C’ by reflecting ΔA’B’C’ over two concurrent lines. (This is a rotation).

Three-Reflections Case 2

Statement	Reason
2. Let ΔABC be an arbitrary triangle.	Construction
33. Randomly choose ΔA’B’C’ congruent to ΔABC with the opposite orientation.	Construction
34. $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, $\overline{BC} \cong \overline{B'C'}$ ∠ABC ≅ ∠A’B’C’, ∠BAC ≅ ∠B’A’C’, ∠ACB ≅ ∠A’C’B’	ΔA’B’C’≅ ΔABC (2), and corresponding parts of congruent triangles are congruent.
35. Draw $\overline{AA'}$.	Construction
36. Draw f, the perpendicular bisector of $\overline{AA'}$.	Construction
37. Reflection over f maps A’ to A.	f is the perpendicular bisector of $\overline{AA'}$. (5).
38. Reflection over f maps $\overline{A'B'}$ to $\overline{AB''}$.	Reflection over f maps A’ to A (6) and B’ to B’’.
39. Reflection over f maps ∠A’B’C’ to ∠AB’’C’’.	Reflection over f maps A’ to A (6), B’ to B’’, and C’ to C’’.
40. Reflection over f maps $\overline{B'C'}$ to $\overline{B''C''}$.	Reflection over f maps B’ to B’’ and C’ to C’’.
41. Draw $\overline{BB''}$.	Construction
42. Draw h, the perpendicular bisector of $\overline{BB''}$.	Construction
43. Reflection over h maps B’’ to B.	h is the perpendicular bisector of $\overline{BB''}$ (11).
44. $\overline{A'B'} \cong \overline{AB''}$	The reflection over f maps $\overline{A'B'}$ to $\overline{AB''}$ (7), and rigid motions preserve length.
45. $\overline{AB} \cong \overline{AB''}$	$\overline{AB} \cong \overline{A'B'}$ (3), $\overline{A'B'} \cong \overline{AB''}$ (13), and congruence of line segments is transitive.
46. A is on h.	$\overline{AB} \cong \overline{AB''}$ (14) so A is equidistant from B and B’’ by definition of congruence. Thus, A is on $\overline{BB''}$’s perpendicular bisector
47. Reflection over h maps A to A.	A is on h (15).
48. Reflection over h maps ∠AB’’C’’ to ∠ABC’’’.	Reflection over h maps A to A (16), B’’ to B (12), and C’’ to C’’’.
49. Reflection over h maps $\overline{AB''}$ to $\overline{AB}$.	Reflection over h maps A to A (16) and maps B’’ to B (12).
50. Reflection over h maps $\overline{B''C''}$ to $\overline{BC'''}$.	Reflection over h maps B’’ to B (12) and C’’ to C’’’.
51. Let j be the line containing A and B.	Construction
52. $\overline{AB}$ is on j.	A and B are on j (20).
53. Reflecting over j maps $\overline{AB}$ to $\overline{AB}$	$\overline{AB}$ is on j.

54. The composition of reflections over f and h maps ∠A'B'C' to ∠ABC'''.	Reflection over f maps ∠A'B'C' to ∠AB''C'' (8), and reflection over h maps ∠AB''C'' to ∠ABC''' (17).
55. ∠A'B'C' ≅ ∠ABC'''	The composition of reflections over f and h maps ∠A'B'C' to ∠ABC''' (23), and rigid motions preserve angle measure.
56. ∠ABC ≅ ∠ABC'''	∠ABC ≅ ∠A'B'C'(3), ∠A'B'C' ≅ ∠ABC'''(24), and congruence of angles is transitive.
57. j is the angle bisector of the angle formed by the rays $\overrightarrow{BC'''}$ to $\overrightarrow{BC}$.	∠ABC ≅ ∠ABC''' (25) , $\overline{AB}$ is on j (21), and $\overrightarrow{BC'''}$ and $\overrightarrow{BC}$ are on opposite sides of j since the orientation of the triangles is opposite after two reflections.
58. Reflection over j maps $\overrightarrow{BC'''}$ to $\overrightarrow{BC}$.	j is the angle bisector of the angle formed by the rays $\overrightarrow{BC'''}$ to $\overrightarrow{BC}$.
59. The composition of reflections over f and h maps $\overline{B'C'}$ to $\overline{BC'''}$.	Reflection over f maps $\overline{B'C'}$ to $\overline{B''C''}$ (9), and reflection over h maps $\overline{B''C''}$ to $\overline{BC'''}$ (19).
60. $\overline{B'C'} \cong \overline{BC'''}$	The composition of reflections over f and h maps $\overline{B'C'}$ to $\overline{BC'''}$(28), and rigid motions preserve length.
61. $\overline{BC} \cong \overline{BC'''}$	$\overline{BC} \cong \overline{B'C'}$ (3), $\overline{B'C'} \cong \overline{BC'''}$(29), and congruence of line segments is transitive.
62. C''' maps to C.	The image of C''' under reflection over j must be on $\overrightarrow{BC}$ (27), and it must be a distance $\lvert\overline{BC'''}\rvert = \lvert\overline{BC}\rvert$ from B (30), which corresponds to the point C.
63. For any two congruent triangles with the opposite orientation, one triangle can be mapped to the other under a composition of three reflections.	ΔABC is an arbitrary triangle and ΔA'B'C' is randomly chosen such that it is congruent to ΔABC with the opposite orientation. We obtain ΔABC from ΔA'B'C' by reflecting ΔA'B'C' over three lines that are not all parallel or all concurrent. (This is a glide reflection).

*For teacher: see **10_random.ggb** for an explanation of why we only get rotations or glide reflections in the three-reflections theorem.

Lesson 5:

Triangle Congruence

Objective: Students will apply transformations to proving SAS, AAS, SSS, and ASA using GeoGebra.

Assessment: Written response quiz developing a formal statement-reason proof using previously developed theorems.

Colorado State Standards: 4.1 b. ii, iii, iv, c. i, ii

Contents:

- **Notes 5.1**
- **In Class Exploration 5.1**
- **Quiz 5.1**

Notes 5.1

Congruence – A transformation of the plane is called a congruence if it is a composition of a finite number of basic isometries.

All of the transformations we have learned so far are isometries so congruence is any composition of the transformations we have learned so far.

In Class Exploration 5.1

In class we will go through the scripted GeoGebra proofs of ASA and SAS. In this way the students will see step by step how and why these isometries define congruence.

ASA

Given: ΔABC with ∠A and ∠B and segment AB measured.

Prove: any triangle with angle measure equal to ∠A and ∠B and a side with length AB arranged in the same order as in ΔABC is congruent to ΔABC.

1. ΔABC with ∠A and ∠B and segment AB measured	Given
2. Make segment A’B’ with rays of angle measure ∠A’ and ∠B’ coming from the respective points	Construction
3. Label the point that these rays intersect as point C’	Construction
4. Label this triangle A’B’C’	Construction
5. Translate ΔA’B’C’ along the vector from A’ to A	Definition Translation
6. Measure the angle from B’AB	Protractor postulate
7. Rotate our new image of ΔA’B’C’ around A by this angle	Definition Rotation
8. Reflect this image over the line AB	Definition of Reflection

Use 11_ASA.ggb

We start with a given triangle ΔABC and we can measure ∠A and ∠B and segment AB. Now we can begin to construct another triangle which has an angle with the same measure as ∠A, a side with the same length as AB and also an angle with the same measure as ∠B. We can construct rays along the angles we just created. Where these intersect we will call C' and construct a triangle ΔA'B'C' with these new points. Now that we have constructed these two triangles we can begin to perform isometries on them in order to show their congruence. We start with translating ΔA'B'C' along the vector from A' to A. Now we rotate this triangle by ∠BAB' so that line A'B' is lying on top of AB. Now we can reflect this triangle across the line AB, this will map ΔA'B'C' directly onto ΔABC. Since we know that ∠A = ∠A', and that ∠B = ∠B' We know that the $\overrightarrow{AC} = \overrightarrow{AC'}$ also that $\overrightarrow{BC} = \overrightarrow{BC'}$ because $\overrightarrow{AC} \cap \overrightarrow{BC} = \overrightarrow{AC'} \cap \overrightarrow{BC'}$ This shows that C=C'. It is now easy to see that under the composition of rigid isometries denoted by ϑ(), that ϑ(ΔA'B'C') = ΔABC and ϑ(ϑ(ΔA'B'C')) = ΔA'B'C'. Because of the definition of an isometry we know that ΔA'B'C' ≅ ΔABC.

SAS

(note: this is specific to this example but would work regardless of what lengths are given)

Given: segments AC and AB, also ∠A

Prove: any triangle with angle measure equal to ∠A and a side with length AB and AC arranged in the same order as in ΔABC is congruent to ΔABC.

1. ΔABC with ∠A and segments AB and AC measured	Given
2. Make segment A'B' with ∠A' with the same measure as AB and ∠A	Construction
3. Draw a circle with radius AC around A' and mark where this intersects ∠A', mark this point as C'	Construction
4. Translate ΔA'B'C' along the vector from A' to A	Definition Translation
5. Measure the angle B'AB	Protractor postulate
6. Rotate the image of ΔA'B'C' around A by the angle we just measured	Definition Rotation
7. Reflect this new image over AB	Definition Reflection

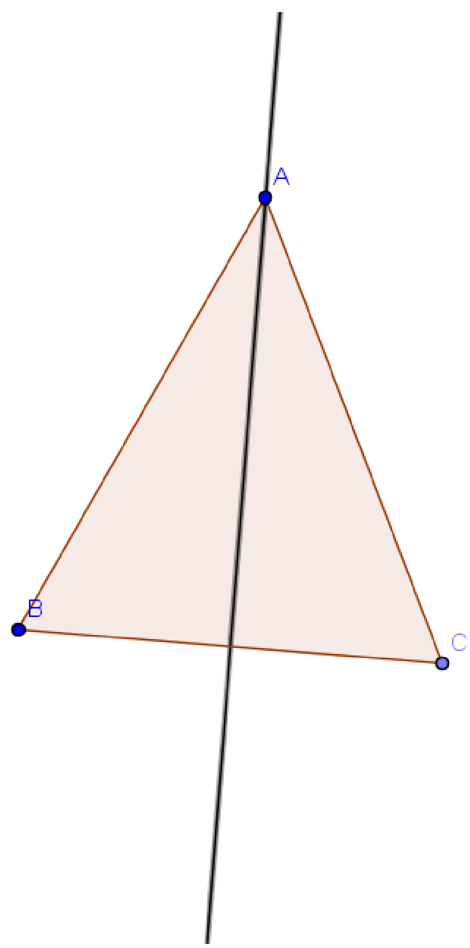
Use 12_SAS.ggb

We are given ΔABC with segment AC = 5.11, AB = 4.42, and ∠A = 97.57 From this information we will construct ΔA'B'C' with A'C' = AC, A'B' = AB, and ∠A' = ∠A. We will start by constructing A'C'. Next we will construct ∠A' which is equal to ∠A Now we will measure a length equal to AB and draw a circle centered at A' and see where it intersects ∠A'. Mark this point as B'. In this way A'B' = AB. Now construct Δ A'B'C' using the points A', B', and C' that we have constructed. Now we will translate ΔA'B'C' using the vector from A' to A. Now we will rotate this triangle around the point A by the ∠B'AB. Now we will reflect this image over the line AB. We know $\overrightarrow{AC} = \overrightarrow{A'C'}$ because ∠BAC = ∠B'A'C'. Since AC = A'C' and they lie on the same ray, and start at the same point, A, we can see that C = C'. We know now that A' is mapped directly to A, B' is mapped to B because AB = A'B' and C' is mapped to C since AC = A'C'. Because these transformations map each point of ΔA'B'C' to the respective points of ΔABC we know that ΔA'B'C' = ΔABC.

Quiz 5.1

1. Give the definition of congruence.

2. Using what is learned in class, prove that the base angles of an isosceles triangle are congruent when already given that they need to construct an angle bisector of the top angle.



Transformational Geometry Project

The goal of this project is to help gain a better understanding of the types of transformations, and to explore where this topic is relevant and present in the world around us, specifically in a field which is interesting to you. This project will be presented as a five paragraph essay.

Audience: You should be writing to your peers that are not in this class, helping explain to them some places where geometry is present in your life.

Format: Your findings should be presented in a five paragraph essay; *typed, double spaced, appropriate header, of 1000-1200 words*

Topic: You have two choices for this project:

- *Using your own camera (or one borrowed from the school library), find an example of each of the three transformations we have learned about in class (translation, reflection, rotation). You may substitute one of these for a glide reflection if you wish.*
- *Using any materials you wish (paint, sculpture, knitted project, poem, etc.) create an art project inspired by transformational geometry. You must use examples of each type of transformation we have talked about in class (translation, rotation, reflection) in your piece.*

Based on the examples you found (either in everyday life or in your art piece), write a five paragraph essay explaining your findings. The first paragraph should be an introduction, including a thesis statement which is appropriate to your examples. The second, third, and fourth paragraphs should be body paragraphs, each one explaining one type of transformation and its example. In these paragraphs, you should define the transformation, and explain how that transformation applies to the image given. An analysis should also be given of why the transformation works here, or how it is used to create functionality or beauty in your image. The fifth paragraph should be a conclusion paragraph, which includes a reworded thesis statement.

In addition to including your images or your work of art either within your essay or as appendices at the end of your paper, *representations of those images created in GeoGebra* should also be included. For this component, draw using the "polygon" tool the shape that you have found or created. Then, by defining either a point to rotate about, a vector to translate by, or a line of reflection, transform the initial polygon to recreate the image you found or created. If you need help with GeoGebra, feel free to use online resources, or ask the teacher. Images created on GeoGebra can be exported or "print-screened" in order to include them in your Word document.

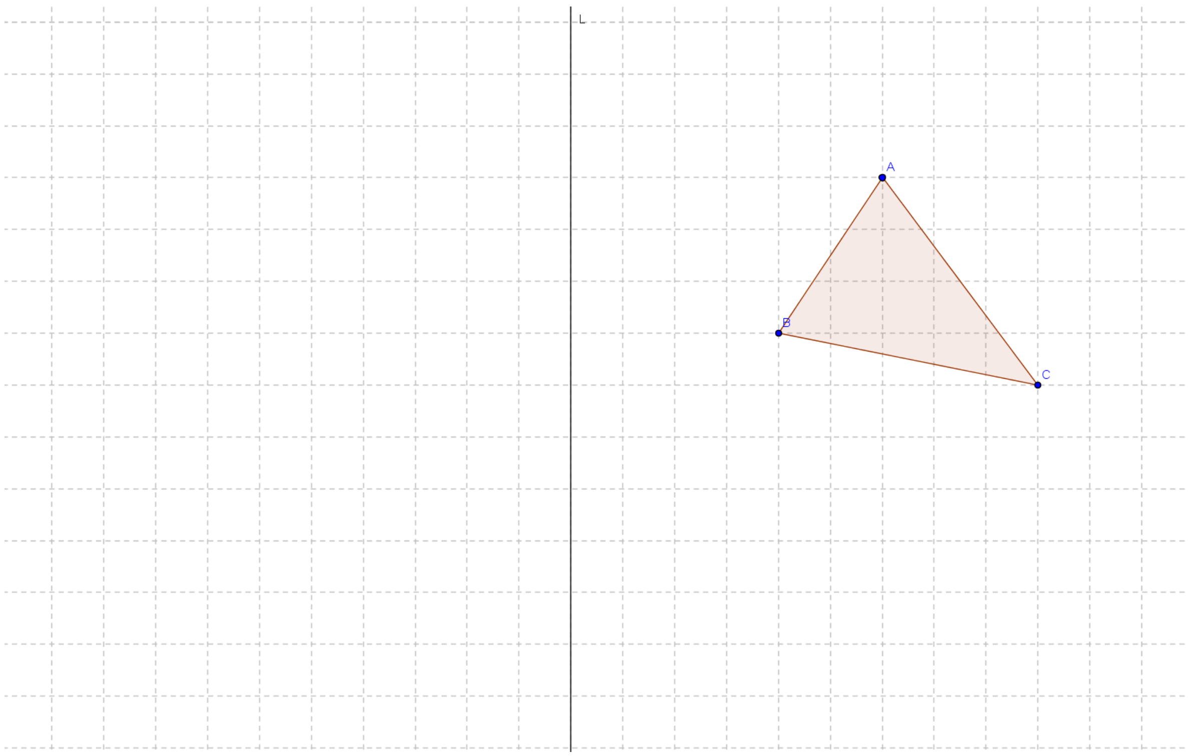
If online or text (or any other type) of resources are used, *cite those sources in MLA format*. All examples/evidence must be self-created. That is, you must have taken the picture yourself or created the work of art yourself. If you have an idea for a project outside these boundaries, please discuss with the teacher before beginning work.

*****Please review the rubric as well as these instructions before beginning your project. Notice that presentation and grammar are just as important as your mathematical findings.***

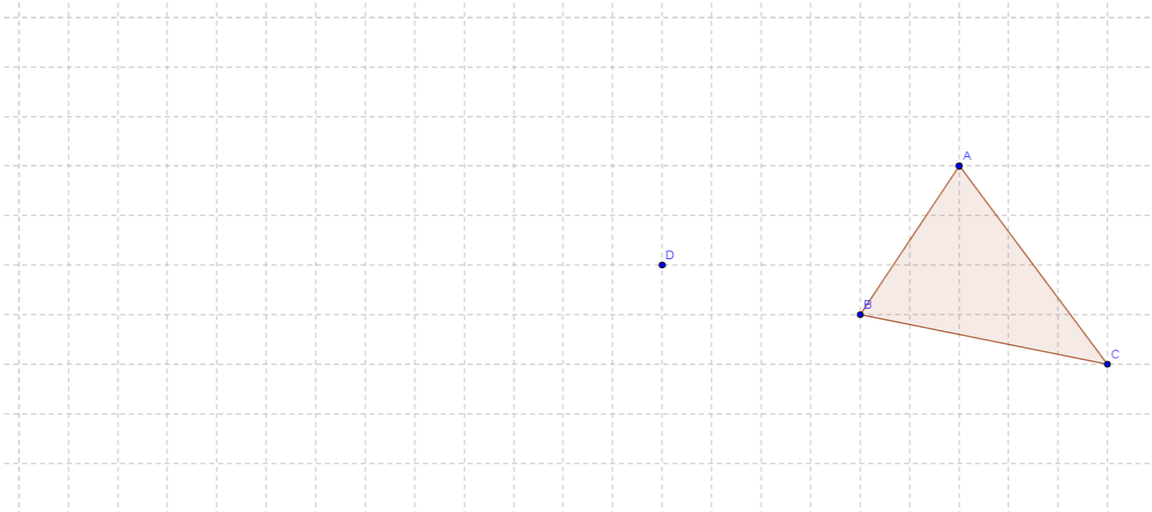
	Organization	Evidence	Technology	Arguments	Mechanics
4	Paper includes introduction, conclusion, and three body paragraphs. Introduction includes a relevant thesis, and conclusion includes reworded thesis. Each body paragraph is organized appropriately.	Examples were self-created/found. Evidence is provided very cleanly and clearly, and supports arguments. Any sources used are cited correctly.	GeoGebra representations accurately reflect given evidence. Transformation is clearly shown and labeled.	Easy to follow and show a deep, mature understanding of transformations. Arguments strongly support thesis.	Spelling and grammar are perfect. Paper is appropriate length and formatted correctly.
3	Paper includes introduction, conclusion and three body paragraphs. Introduction includes relevant thesis. Flow overall is weaker, but arguments support thesis.	Examples were self-created/found. Evidence is presented in an organized manner, and mostly supports arguments. Any sources used are cited correctly.	GeoGebra representations mostly accurately reflect given evidence. Transformation is clearly shown and labeled.	Easy to follow and show a strong understanding of transformations. Arguments support thesis.	Spelling and grammar are almost perfect. Errors take away very little from overall meaning. Paper is appropriate length and formatted correctly.
2	Paper includes introduction, conclusion, and three body paragraphs. Flow between sentences or paragraphs is interrupted at times, and some pieces are irrelevant.	Examples were self-created/found. Evidence is presented somewhat clearly, and somewhat supports arguments. Any sources used are cited mostly correctly.	GeoGebra representations somewhat represent given evidence. Transformation is hard to see, or locate labels.	Some claims don’t support thesis or are flawed and incorrect. Show a questionable understanding of transformations.	Mistakes are regular in spelling or grammar. Errors are noticeable throughout paper. Paper is appropriate length and formatted mostly correctly.
1	Paper is missing one or more of the component paragraphs. Subtopics do not support thesis.	Examples were self-created/found. Evidence is hard to see or difficult to find, and only slightly relates to arguments. Sources used but not cited.	GeoGebra representations slightly represent given evidence. Transformation is not shown, or not labeled.	Some or most claims include major flaws and are hard to follow. Slight understanding of transformations.	Flow of paper strongly affected by errors in spelling and grammar. Paper is appropriate length and formatted mostly correctly.
0	Paper is missing one or more of the component paragraphs. Paper does not have flow.	Examples were not self-created/found. Evidence is not present or is not clear and does not relate to arguments. Sources used but not cited.	GeoGebra representations are not present or do not represent given evidence. Transformation is not present, or not labeled.	No clear understanding of transformations.	Errors in spelling and grammar are so frequent that it is difficult to understand the meaning. Paper is not an appropriate length and is formatted incorrectly.

Transformational Geometry Unit Test

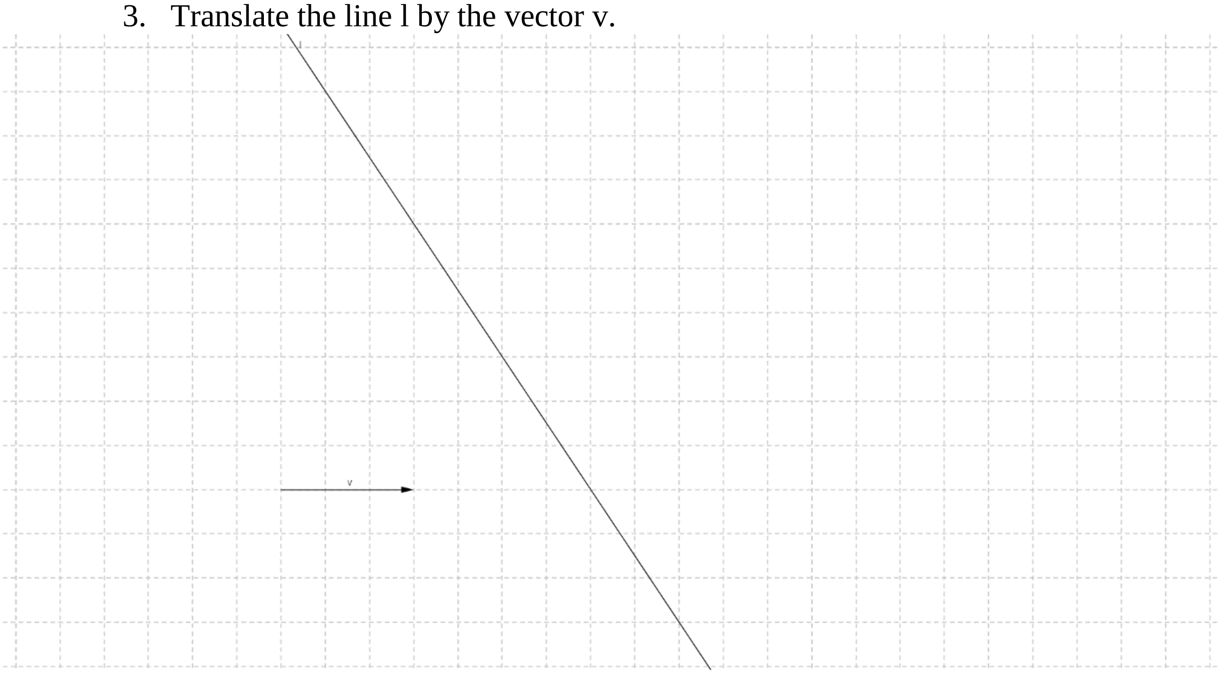
1. Reflect triangle ABC over the line l.



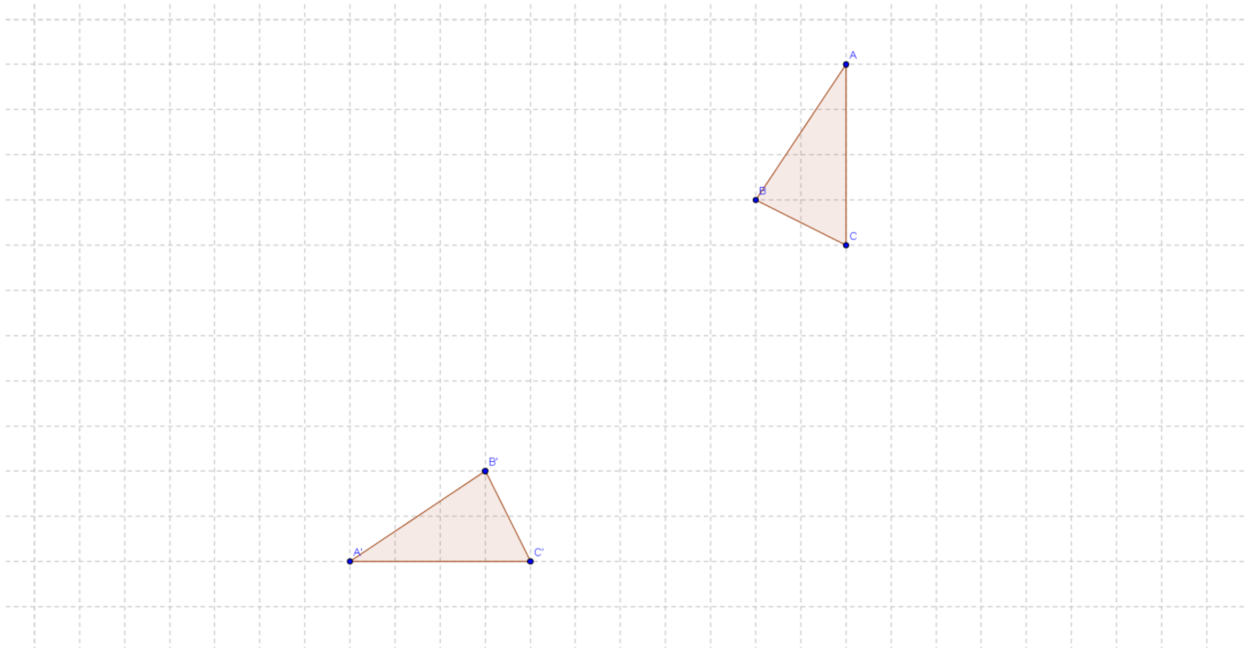
2. Rotate triangle ABC by 180° about the point D.



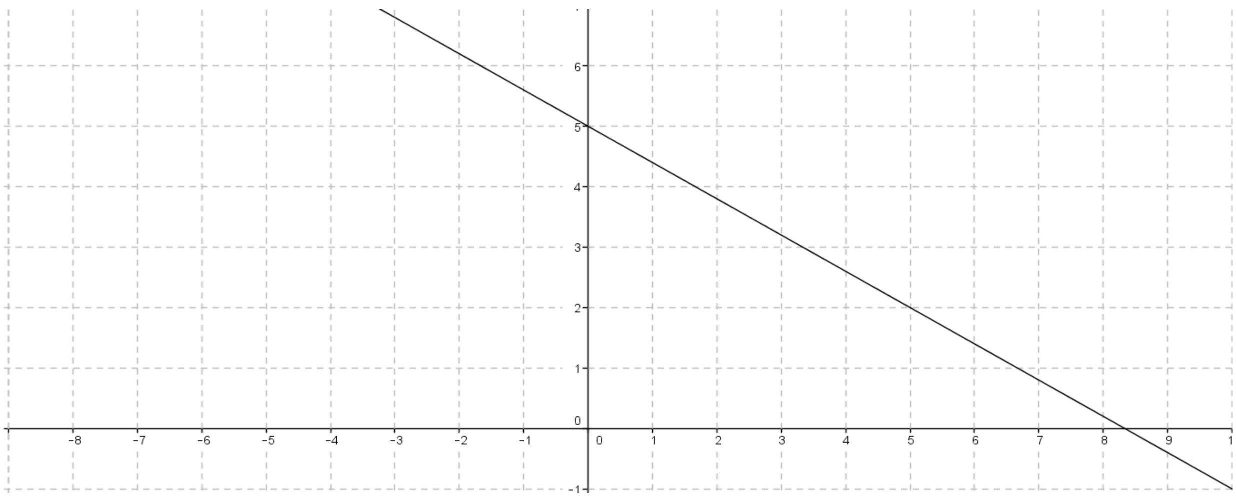
3. Translate the line l by the vector v.



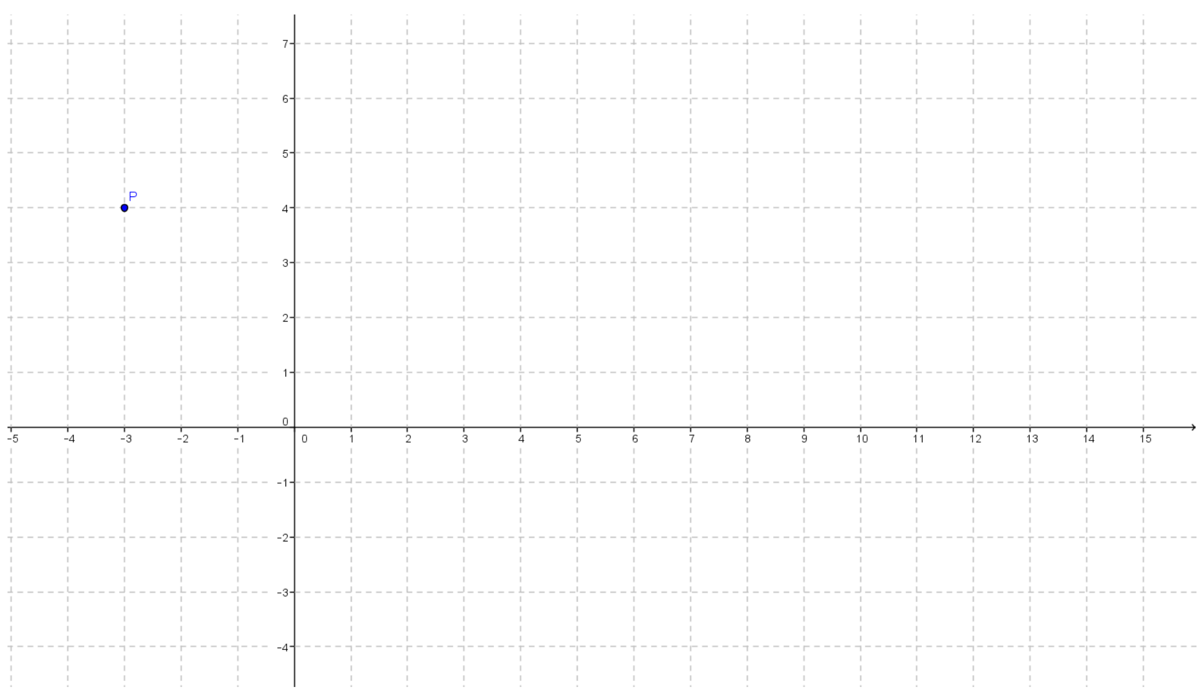
4. Let triangle A’B’C’ be the image of triangle ABC under some transformation. What kind of transformation was applied? Describe how you know.



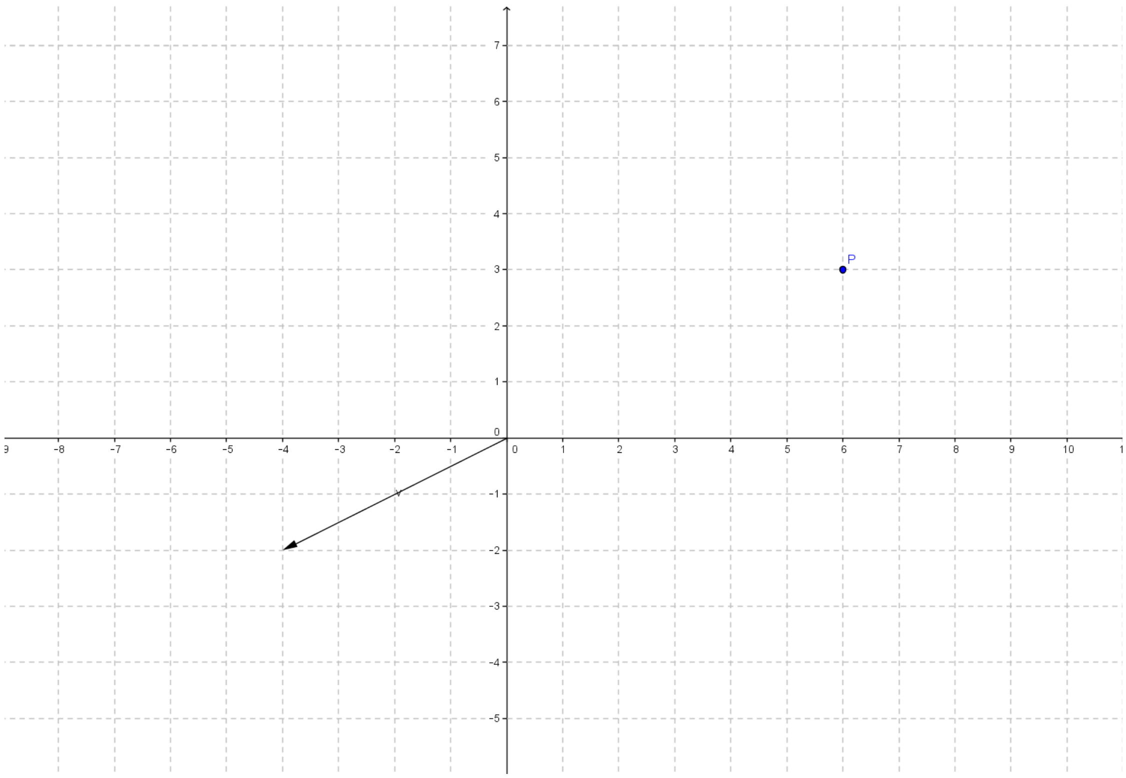
5. Let σ be the reflection over the line l. Give the coordinates of a point P such that σ(P)=P.



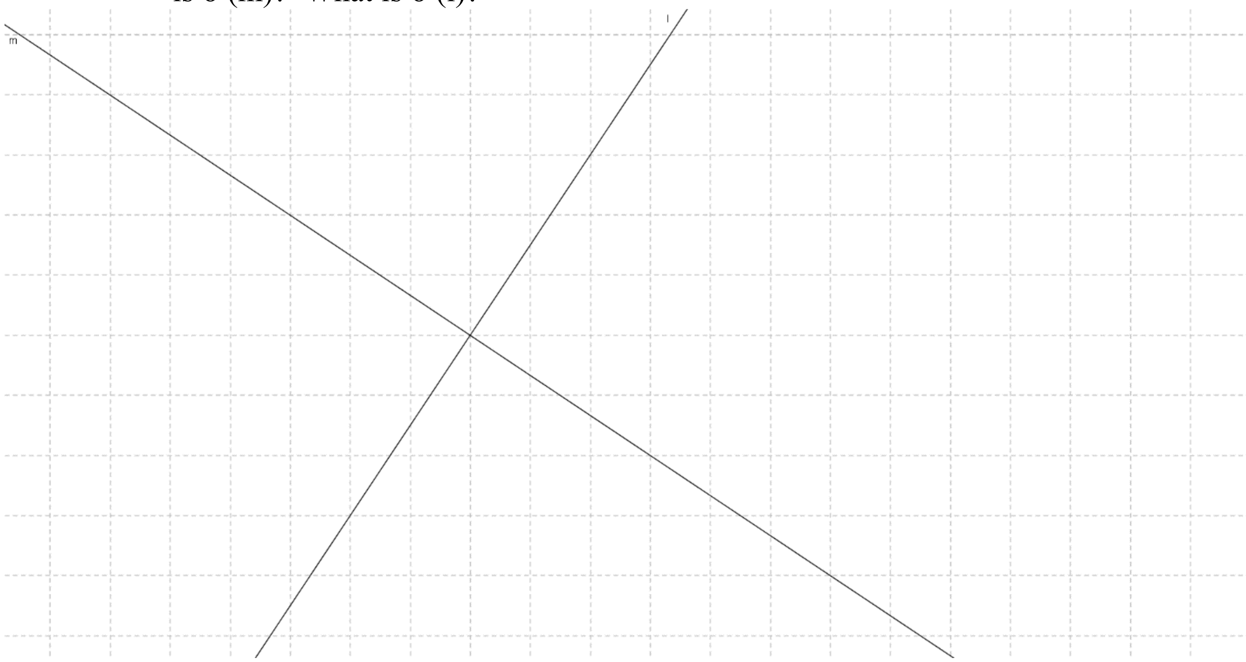
6. Let ρ be the rotation about the origin by 90°. Consider the point P=(-3,4). What is ρ(P)?



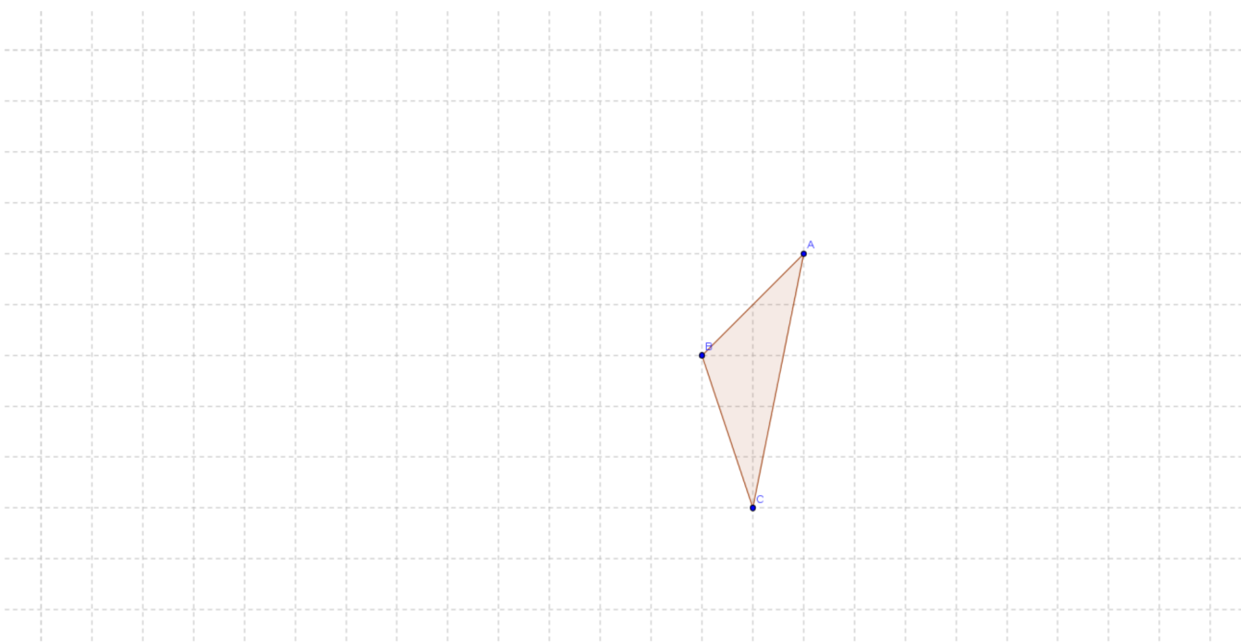
7. Let ω be the translation along the vector v. Consider the point P=(6,3). What is ω (P)?



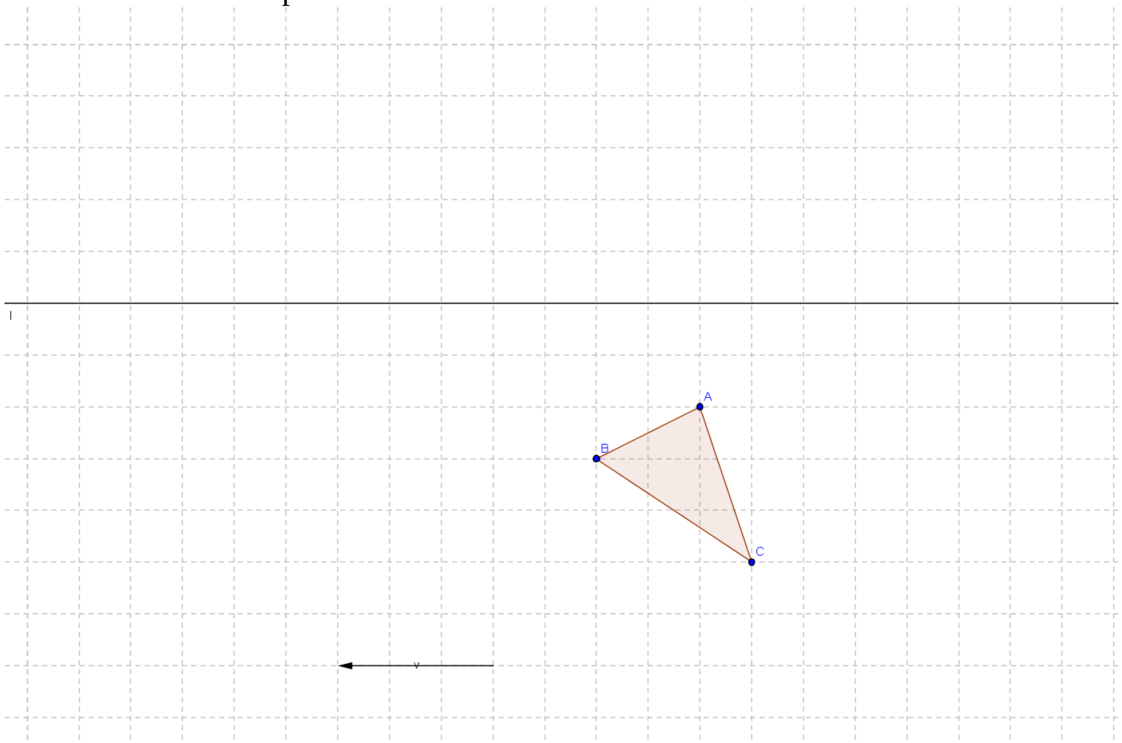
8. Let σ be the reflection over the line l. Consider the line m perpendicular to l. What is σ (m)? What is σ (l)?



9. Consider triangle ABC. Perform a composition of two transformations of your choice to obtain some triangle A'B'C'. Describe the two transformations you chose, using appropriate function notation.



10. Consider the triangle ABC. Draw triangle A'B'C', the image of triangle ABC under the translation by the vector v. Next, draw the triangle A"B"C", the image of triangle A'B'C' under the reflection over the line l. What kind of transformation is this composition of transformations?



11. Fill in the blanks.

Let σ be a composition of reflections, and let the image of triangle ABC under σ be A'B'C'. If the orientation of ABC and A'B'C' is:

the same, then σ is a composition of an _______number of reflections.

the opposite, then σ is a composition of an ________ number of reflections.

For 12 and 13, choose the correct response.

12. If α is the composition of the reflections in the lines l and m, and α is a translation, what must be true of lines l and m?
    a) l and m are parallel
    b) l and m are perpendicular
    c) l and m are concurrent

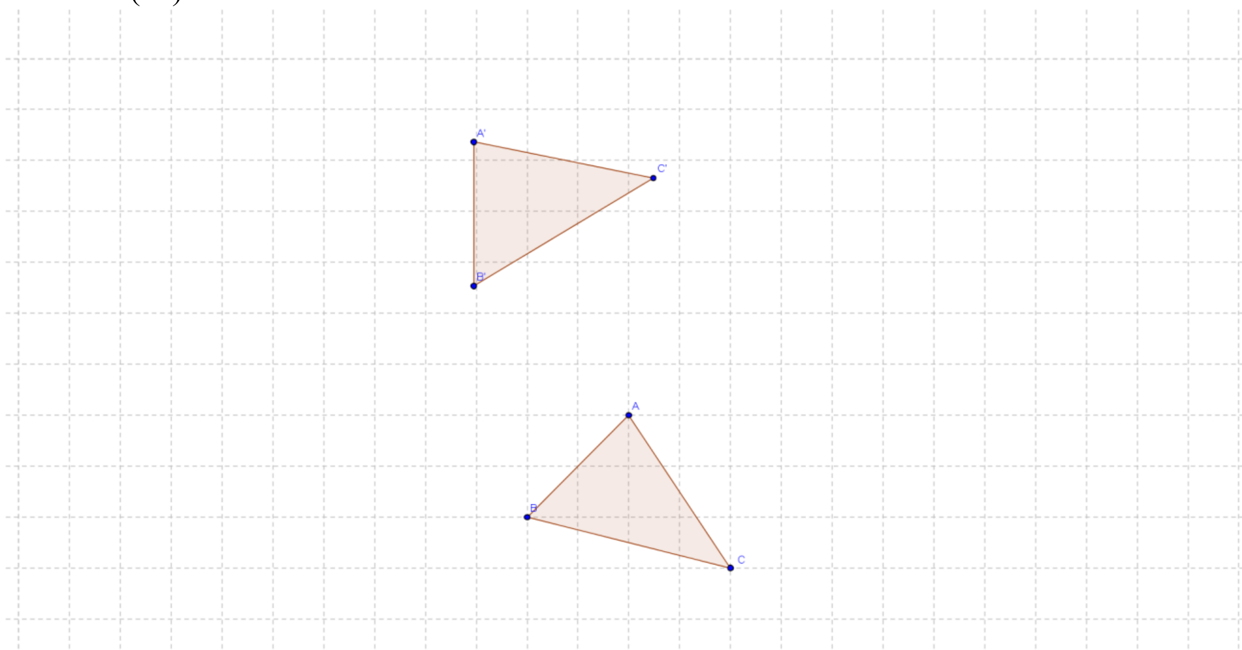
13. If β is the composition of the reflections over the lines l and m, and β is a rotation, what must be true of the lines l and m?
    a) l and m are parallel
    b) l and m are perpendicular
    c) l and m are concurrent

14. State the three reflection theorem.

15. a. Consider triangles ABC and A'B'C' with ABC = A'B'C'. Perform a reflection such that the image of the point A' under the reflection is A. Call the image of this reflection triangle A"B"C".

b. Consider triangles ABC and A"B"C". Perform a reflection such that the image of the point B" under the reflection is B.

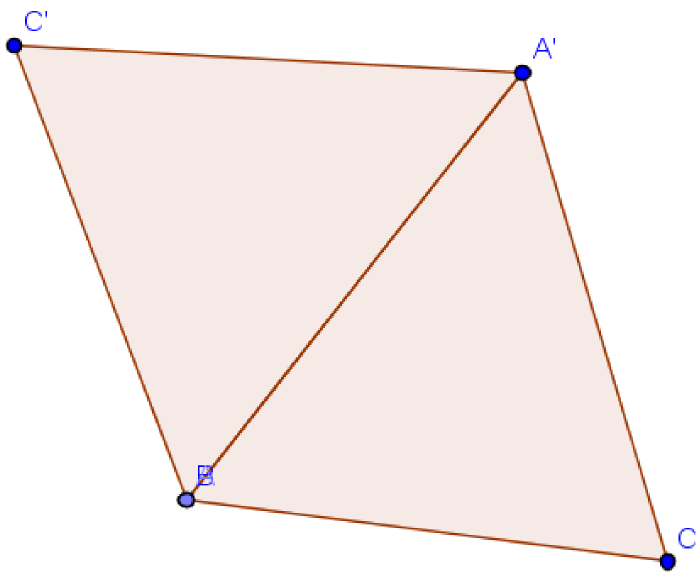
c. Let σ be the composition of the two reflections you performed in parts a and b. What is $\sigma(C')$?



16. Given: two triangle with all three sides of equal length

Prove: that these triangles are congruent

Hint: draw in segment connecting C to C’ and use what we know about isosceles triangles and triangle congruence to show that these two triangles are congruent.



References

Burger, Edward B., et al. *Geometry*. U.S.A.: Houghton Mifflin Harcourt, 2010. Print.

Wu, H. *Pre-Algebra*. 2010.

Developers

Rebecca Cooper: Lesson 3, Lesson 4

Charles Grossen: Lesson 1, Lesson 5

Sarah Nelson: Lesson 2, Unit Project